Teichmüller spaces for geometric structures, lecture 1

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Plan:

1. Set-up: Teichmüller space and the moduli space of geometric structures.

2. Moser's theorem. Teichmüller space of symplectic structures on a torus.

3. Hyperkähler maifolds (introduction)

Geometric structures

DEFINITION: "Geometric structure" on a manifold M is a reduction of its structure group $GL(n,\mathbb{R})$ to a subgroup $G \subset GL(n,\mathbb{R})$. However, it is easier to define it by a collection of tensors $\Psi_1,...,\Psi_n$ such that the stabilizer $\operatorname{St}_{\langle\Psi_1,...,\Psi_n\rangle}$ of $\Psi_1,...,\Psi_n$ at each point $x \in M$ is conjugate to the same group $G \subset GL(d,\mathbb{R})$, $d = \dim_{\mathbb{R}} M$.

Let me give some examples.

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: Symplectic form on a manifold is a non-degenerate differential 2-form ω satisfying $d\omega = 0$.

Teichmüller space of geometric structures

Let $\mathcal{C}$ be the set of all geometric structures of a given type, say, complex, or symplectic. We put topology of uniform convergence with all derivatives on $\mathcal{C}$. Let $\operatorname{Diff}_0(M)$ be the connected component of its diffeomorphism group $\operatorname{Diff}(M)$ (**the group of isotopies**).

DEFINITION: The quotient $\mathcal{C}/\operatorname{Diff}_0$ is called **Teichmüller space** of geometric strictures of this type.

DEFINITION: The group $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ is called **the mapping class group** of M. It acts on Teich by homeomorphisms.

DEFINITION: The orbit space $\mathcal{C}/\operatorname{Diff} = \operatorname{Teich}/\Gamma$ is called **the moduli space** of geometric structure of this type.

Teichmüller space for symplectic structures

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. **Then $\Gamma(\Lambda^2 M)$ is a Frechet vector space, and Symp a Frechet manifold.**

DEFINITION: Consider the group of diffeomorphisms, denoted Diff or $\operatorname{Diff}(M)$ as a Frechet Lie group, and denote its connected component ("group of isotopies") by Diff_0. The quotient group $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ is called **the mapping class group** of M.

DEFINITION: **Teichmüller space of symplectic structures on M** is defined as a quotient $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$. The quotient $\operatorname{Teich}_s / \Gamma = \operatorname{Symp} / \operatorname{Diff}$, is called **the moduli space of symplectic structures.**

REMARK: In many cases Γ acts on Teich_s with dense orbits, hence **the moduli space is not always well defined.**

DEFINITION: Two symplectic structures are called **isotopic** if they lie in the same orbit of Diff_0, and **diffeomorphic** is they lie in the same orbit of Diff.

Moser's theorem

DEFINITION: Define **the period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

Theorem 1: (Moser)
Let ω_t, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold S. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are diffeomorphic.**

The proof of Moser's theorem

THEOREM: (Moser)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

Proof. Step 1: We can locally find a section S for the Diff_0-action on Symp, producing a local decomposition $\operatorname{Symp} = O \times S$, where O is a Diff_0-orbit. Here O and S are both Frechet manifolds.

Step 2: The period map $P: S \longrightarrow H^2(M, \mathbb{R})$ is a smooth submersion. By Theorem 1, the fibers of P are 0-dimensional. Therefore, P is locally a diffeomorphism. ■

Symplectic structures on a compact torus

DEFINITION: A symplectic structure ω on a torus is called **standard** if there exists a flat torsion-free connection preserving ω.

REMARK: Moser's theorem immediately implies that **the set** Teich_{st} **of standard symplectic structures is open in the Teichmüller space.** Indeed, the period map from Teich_{st} to $H^2(M)$ is also locally a diffeomorphism.

REMARK: It is not known if any non-standard symplectic structures exist (even in dimension =4).

THEOREM: Let $\Lambda^2_{nd}(H_1(T)) \subset H^2(T)$ be the space of symplectic forms on $H_1(T)$, where T is an even-dimensional torus. Consider the period map $\mathrm{Per}: \mathrm{Teich}_{st} \longrightarrow \Lambda^2_{nd}(H_1(T)) \subset H^2(T)$, where Teich_{st} is the Teichmüller space of standard symplectic structures on T. **Then** Per **is a diffeomorphism.**

Non-Hausdorff points on symplectic Teichmüller space

Example of D. McDuff found in Salamon, Dietmar, *Uniqueness of symplectic structures*, Acta Math. Vietnam. 38 (2013), no. 1, 123-144.

Let $M = S^1 \times S^1 \times S^2 \times S^2$ with coordinates $\theta_1, \theta_2 \in S^1 \subset \mathbb{C}^*$ and $z_1, z_2 \in S^2$. Let $\varphi_{\theta,z}$ $\mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a rotation around the axis $z \in \mathbb{C}P^1$ by the angle θ. **Consider the diffeomorphism $\Psi : M \longrightarrow M$ mapping $(\theta_1, \theta_2, z_1, z_2)$ to $(\theta_1, \theta_2, z_1, \varphi_{\theta_1, z_1}(z_2))$.**

THEOREM: Let ω_λ be the product symplectic form on $M = T^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ obtained as a product of symplectic forms of volume 1, 1, λ on T^2, $\mathbb{C}P^1$, $\mathbb{C}P^1$. **The form $\Psi^*(\omega_1)$ is homologous, but not diffeomorphic to ω_1.** However, **the form $\Psi^*(\omega_\lambda)$ is diffeomorphic to ω_λ for any $\lambda \neq 1$.**

(D. McDuff, *Examples of symplectic structures*, Invent. Math. 89 (1987), 13-36.)

The space of standard symplectic forms on a torus

THEOREM: Let $\Lambda^2_{nd}(H_1(T)) \subset H^2(T)$ be the space of symplectic forms on $H_1(T)$, where T is an even-dimensional torus. Consider the period map $\operatorname{Per}: \operatorname{Teich}_{st} \longrightarrow \Lambda^2_{nd}(H_1(T)) \subset H^2(T)$, where $\operatorname{Teich}_{st}$ is the Teichmüller space of standard symplectic structures on T. **Then Per is a diffeomorphism.**

Proof. Step 1: Let Teich_h be the Teichmüller space of flat Hermitian metrics on T. Clearly, $\operatorname{Teich}_h = GL(2n, \mathbb{R})/U(n)$. Moreover, **the natural forgetful map $\operatorname{Teich}_h \longrightarrow \operatorname{Teich}_{st}$ is surjective**.

Step 2: The fibers of the natural projection $\operatorname{Teich}_h \longrightarrow \Lambda^2_{nd}(H_1(T))$ are connected. Using the diagram

$$\begin{array}{ccc} \operatorname{Teich}_h & \longrightarrow & \Lambda^2_{nd}(H_1(T)) \\ \downarrow & & \downarrow \scriptstyle{\operatorname{Id}} \\ \operatorname{Teich}_{st} & \longrightarrow & \Lambda^2_{nd}(H_1(T)) \end{array}$$

we obtain that $\operatorname{Per}: \operatorname{Teich}_{st} \longrightarrow \Lambda^2_{nd}(H_1(T))$ has connected fibers. By Moser's theorem, this map is a diffeomorphism. ■

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

Kähler manifolds

DEFINITION: A Riemannian metric g on a complex manifold (M, I) is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

REMARK: This is equivalent to $\nabla\omega = 0$, where ∇ is Levi-Civita connection.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

CLAIM: A compact hyperkähler manifold M **has maximal holonomy of Levi-Civita connection $Sp(n)$** if and only if $\pi_1(M) = 0$, $h^{2,0}(M) = 1$.

THEOREM: (Bogomolov decomposition)
Any compact hyperkähler manifold has a finite covering isometric to a product of a torus and several maximal holonomy hyperkähler manifolds.

Holomorphically symplectic manifolds

DEFINITION: **A holomorphically symplectic manifold** is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: In these lectures , all holomorphically symplectic manifolds are assumed to be Kähler and compact.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot,\cdot)$, $\omega_J := g(J\cdot,\cdot)$, $\omega_K := g(K\cdot,\cdot)$.

CLAIM: In these assumptions, $\omega_J + \sqrt{-1}\,\omega_K$ is holomorphic symplectic on (M,I).

THEOREM: **(Calabi-Yau)**
A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of these lectures, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold of maximal holonomy.**

EXAMPLES.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: A non-compact example: $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity** $\mathbb{C}^2/\pm 1$**.**

EXAMPLE: Take a 2-dimensional complex torus T, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

REMARK: Take a symmetric square $\operatorname{Sym}^2 T$, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. **Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $\widetilde{T/\pm 1}$.**

DEFINITION: A complex surface is called **K3 surface** if it a deformation of the Kummer surface.

THEOREM: (a special case of Enriques-Kodaira classification) Let M be a compact complex surface which is hyperkähler. **Then M is either a torus or a K3 surface.**

Hilbert schemes

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Fujiki, Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLE: A Hilbert scheme of K3 is hyperkähler.

EXAMPLE: Let T be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, a universal covering of $T^{[n]}/T$ is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known compact hyperkaehler manifolds are these 2 and the two series:** Hilbert schemes of K3, and generalized Kummer.

Global Torelli theorem.

THEOREM: Let M be a simple hyperkähler manifold, and $\Gamma := \operatorname{Diff}/\operatorname{Diff}_0$ its mapping class group. Then $H^2(M,\mathbb{Z})$ admits a Γ-invariant, non-degenerate integer quadratic form q such that the natural action of Γ on $H^2(M,\mathbb{Z})$ **induces a homomorphism $\Gamma \longrightarrow SO(H^2(M,\mathbb{Z}),q)$ with finite index and finite kernel.**

REMARK: Suppose that $\varphi: M \longrightarrow M'$ is a bimeromorphic map of Calabi-Yau manifolds. Then the exceptional set of φ has codimension $\geqslant 2$, **hence $H^2(M) = H^2(M')$.**

DEFINITION: Let Teich be the Teichmüller space of complex structures of hyperkähler type on M, and Teich_b the quotient of Teich by an equivalence relation induced by bimeromorphic maps $(M,I) \longrightarrow (M,I')$ inducing identity on $H^2(M)$. Then Teich_b is called **birational Teichmüller space** of M. As we shall see, Teich_b is a smooth, Hausdorff complex manifold.

THEOREM: Consider the space

$$\mathbb{P}\mathrm{er} := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l,\bar{l}) > 0\}$$

and let the **period map** $\operatorname{Per}: \operatorname{Teich} \longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map a complex structure I to a line $H^{2,0}(M,I) \in \mathbb{P}H^2(M,\mathbb{C})$. **Then $\operatorname{Per}: \operatorname{Teich} \longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ induces a bijective map $\operatorname{Teich}_b \longrightarrow \mathbb{P}\mathrm{er}$** on any connected component of Teich_b.