Teichmüller spaces for geometric structures, lecture 2

Misha Verbitsky

Conference - Teichmüller Theory in Higher Dimension and Mirror Symmetry

April 24-28, 2017

Plan:

1. Hyperkähler manifolds (introduction)

2. Supersymmetry on Kähler and hyperkähler manifolds

3. Automorphisms of cohomology

4. Computation of the mapping class group

Hyperkähler manifolds (reminder)

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

CLAIM: A compact hyperkähler manifold M **has maximal holonomy of Levi-Civita connection $Sp(n)$ if and only if $\pi_1(M) = 0$, $h^{2,0}(M) = 1$.**

THEOREM: (Bogomolov decomposition)
Any compact hyperkähler manifold has a finite covering isometric to a product of a torus and several maximal holonomy hyperkähler manifolds.

Holomorphically symplectic manifolds (reminder)

DEFINITION: **A holomorphically symplectic manifold** is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: In these lectures, all holomorphically symplectic manifolds are assumed to be Kähler and compact.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot,\cdot)$, $\omega_J := g(J\cdot,\cdot)$, $\omega_K := g(K\cdot,\cdot)$.

CLAIM: In these assumptions, $\omega_J + \sqrt{-1}\,\omega_K$ is holomorphic symplectic on (M,I).

THEOREM: **(Calabi-Yau)**
A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of these lectures, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold of maximal holonomy.**

EXAMPLES.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: A non-compact example: $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity** $\mathbb{C}^2/\pm 1$**.**

EXAMPLE: Take a 2-dimensional complex torus T, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

REMARK: Take a symmetric square $\operatorname{Sym}^2 T$, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. **Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $\widetilde{T/\pm 1}$.**

DEFINITION: A complex surface is called **K3 surface** if it a deformation of the Kummer surface.

THEOREM: (a special case of Enriques-Kodaira classification)
Let M be a compact complex surface which is hyperkähler. **Then M is either a torus or a K3 surface.**

Hilbert schemes

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Fujiki, Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLE: **A Hilbert scheme of K3** is hyperkähler.

EXAMPLE: Let T be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, a universal covering of $T^{[n]}/T$ is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known compact hyperkaehler manifolds are these 2 and the two series:** Hilbert schemes of K3, and generalized Kummer.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.**

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - \frac{n-1}{n} \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n}\right) \left(\int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(b_2 - 3, 3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

Global Torelli theorem.

THEOREM: Let M be a simple hyperkähler manifold, and $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ its mapping class group. Then $H^2(M, \mathbb{Z})$ admits a Γ-invariant, non-degenerate integer quadratic form q such that the natural action of Γ on $H^2(M, \mathbb{Z})$ **induces a homomorphism $\Gamma \longrightarrow SO(H^2(M, \mathbb{Z}), q)$ with finite index and finite kernel.**

REMARK: Suppose that $\varphi: M \longrightarrow M'$ is a bimeromorphic map of Calabi-Yau manifolds. Then the exceptional set of φ has codimension $\geqslant 2$, **hence $H^2(M) = H^2(M')$.**

DEFINITION: Let Teich be the Teichmüller space of complex structures of hyperkähler type on M, and Teich_b the quotient of Teich by an equivalence relation induced by bimeromorphic maps $(M, I) \longrightarrow (M, I')$ inducing identity on $H^2(M)$. Then Teich_b is called **birational Teichmüller space** of M. As we shall see, Teich_b is a smooth, Hausdorff complex manifold.

THEOREM: Consider the space

$$\mathbb{P}\mathrm{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \ | \ \ q(l, l) = 0, q(l, \bar{l}) > 0\}$$

and let the **period map** $\operatorname{Per}: \operatorname{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map a complex structure I to a line $H^{2,0}(M, I) \in \mathbb{P}H^2(M, \mathbb{C})$. **Then $\operatorname{Per}: \operatorname{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ induces a bijective map $\operatorname{Teich}_b \longrightarrow \mathbb{P}\mathrm{er}$** on any connected component of Teich_b.

Automorphisms of cohomology

We are going to prove the following theorem.

Theorem 1: The natural action of Spin$(H^2(M,\mathbb{R}),q)$ on $H^2(M)$ **is extended to an action on the algebra $H^*(M)$ by automorphisms.** Moreover, **this action preserves the Chern classes.**

CLAIM: Let M be a compact Kähler manifold. Consider **the Weil operator** $W \in \operatorname{End}(H^*(M))$ acting on $H^{p,q}(M)$ as $W\Big|_{H^{p,q}(M)} = \sqrt{-1}\,(p-q)$, and let $U(1)$ act on $H^*(M)$ as $e^{-\sqrt{-1}\,W}$. **Then $U(1)$ acts by automorphisms.**

Theorem 1 is implied by the following result, proven later in this lecture.

THEOREM: Consider the Lie subalgebra $\mathfrak{g}_W \subset \operatorname{Aut}(H^*(M))$ generated by the Weil operators W for all Kähler structures of Kähler type. **Then $\mathfrak{g}_W$ is isomorphic to $\mathfrak{so}(H^2(M,\mathbb{R}),q)$.**

Lie superalgebras

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphisms space $\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$ is also graded, with $\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j})$

DEFINITION: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** $\tilde a$ of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde a\tilde b} ba$.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \mathfrak{g}^*\times\mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde a\tilde b}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde a\tilde c}\{a,\{c,b\}\}$

Supersymmetry in Kähler geometry

Let (M, I, g) be a Kaehler manifold, ω its Kaehler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.

1. $L(\alpha) := \omega \wedge \alpha$

2. $\Lambda(\alpha) := *L * \alpha$. It is easily seen that $\Lambda = L^*$.

3. The Weil operator $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$

THEOREM: These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension (5|4)**,** acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence **$\mathfrak{a}$ also acts on the cohomology of M.**

REMARK: This is a convenient way to summarize the Kähler relations and the Lefschetz $\mathfrak{sl}(2)$-action.

Supersymmetry in hyperkähler geometry

Let (M, I, J, K, g) be a hyperkaehler manifold, ω_I, ω_J, ω_K its Kaehler forms. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.

1. $L_I(\alpha) := \omega_I \wedge \alpha$

2. $\Lambda_I(\alpha) := *L_I * \alpha$. It is easily seen that $\Lambda_I = L_J^*$.

3. Three Weil operators $W_I\big|_{\Lambda^{p,q}(M,I)} = \sqrt{-1}(p-q)$, $W_J\big|_{\Lambda^{p,q}(M,J)} = \sqrt{-1}(p-q)$, $W_K\big|_{\Lambda^{p,q}(M,K)} = \sqrt{-1}\,(p-q)$

THEOREM: These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension (11|8), acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence $\mathfrak{a}$ **also acts on the cohomology of M.**

REMARK: The Weil operators form the Lie algebra $\mathfrak{su}(2)$ of unitary quaternions. This means that **the quaternionic action belongs to $\mathfrak{a}$**. In particular, L_J, L_K, Λ_J and Λ_K.

REMARK: The twisted de Rham differentials d_I, d_J, d_K, associated to I, J, K also belong to $\mathfrak{a}$: $d_I = [W_I, d]$, $d_J = [W_J, d]$, $d_K = [W_K, d]$

$\mathfrak{so}(4,1)$-action and the Hodge decomposition

REMARK: 1. $[L_I,\Lambda_J]=W_K$, $[L_J,\Lambda_K]=W_I$, $[L_I,\Lambda_K]=-W_J$.

2. The even part of $\mathfrak{a}$ **is isomorphic to $\mathfrak{sp}(1,1,\mathbb{H})\oplus\mathbb{R}\cdot\Delta$.**

3. The odd part $\langle d, d_I, d_J, d_K, d,^* d_I^*, d_J^*, d_K^*\rangle$ **generates the 9-dimensional odd Heisenberg algebra,** with the only non-trivial supercommutators being $\{d,d^*\}=\{d_I,d_I^*\}=\{d_J,d_J^*\}=\{d_K,d_K^*\}=\Delta$

4. The action of $\mathfrak{a}_{even}$ on $\mathfrak{a}_{odd}$ **is the fundamental representation of $\mathfrak{sp}(1,1,\mathbb{H})$ in $\mathbb{H}^2$,** with the quaternionic Hermitian metric on $\mathfrak{a}_{odd}$ provided by the anticommutator.

COROLLARY: The weight decomposition of the $\mathfrak{sp}(1,1,\mathbb{H})=\mathfrak{so}(4,1)$-action on $H^*(M)$ **coincides with the Hodge decomposition.**

Lia algebra $\mathfrak{g}$ generated by $\mathfrak{sl}(2)$-triples

THEOREM: Let M be a hyperkähler manifold of maximal holonomy, A^* its cohomology algebram and $\mathfrak{g} := \mathfrak{g}(A)$ the Lie algebra generated by all Lefschetz $\mathfrak{sl}(2)$-triples. **Then $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(b_2 - 2, 4)$.**

Sketch of the proof. Step 1: Consider the action of $\mathfrak{g}$ on the **Mukai extension** $\hat{H}^2(M) := \mathbb{R} \cdot x \oplus H^2(M) \oplus \mathbb{R} \cdot y$, where x has grading 0, y has grading 4, $H^2(M)$ has grading 2. We equip $\hat{H}^2(M)$ with **the Mukai form** which is equal to BBF on $H^2(M)$, preserves grading, and satisfies $q_M(x, y) = 1$ $q(x, x) = q(y, y) = 0$, $x, y \perp H^2(M)$ and $(x, y) = 1$. The action of $\mathfrak{g}$ on $\hat{H}^2(M)$ is determined by the following properties: **1. It is compatible with the grading. 2. For all $\alpha, \beta \in H^2(M)$, one has $L_\alpha x = \alpha$, $L_\alpha \beta = q(\alpha, \beta) y$, where q is the BBF form. 3. $\Lambda_\alpha y = \alpha$, $\Lambda_\alpha \beta = q(\alpha, \beta) x$.**

To see that this action is well-defined, we need to check that commutator relations hold. This follows from commutator relations in $\mathfrak{so}(1, 4)$ and Zariski density of pairs $\alpha, \beta \in \langle \omega_I, \omega_J, \omega_K \rangle$ in the set of all pairs $\alpha, \beta \in H^2(M)$.

Step 2: The map $\mathfrak{g} \longrightarrow \mathfrak{so}(\hat{H}^2(M))$ is surjective, which follows from the dimension argument (dimensions are computed using the local Torelli theorem). Injectivity of $\mathfrak{g} \longrightarrow \mathfrak{so}(\hat{H}^2(M))$ is clear, because $\mathfrak{so}(\hat{H}^2(M))$ is given by generators and relations which hold true in $\mathfrak{g}$. ■

Hodge structures and $\mathfrak{g}$-action

REMARK: The Lie algebra $\mathfrak{g} = \mathfrak{so}(b_1 - 2, 4)$ is equipped with a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus g_2$, induced by the grading on the Mukai space: $\hat{H}^2(M) := H_0 \oplus H^2(M) \oplus H_4$, with H_0 and H_4 1-dimensional. Then $\mathfrak{g}_0 = \mathfrak{g}_0' \oplus H$, where $H = [L_\omega, \Lambda_\omega]$ is the operator inducing the grading and commuting with the rest of $\mathfrak{g}_0$, denoted by $\check{\mathfrak{g}}_0$.

REMARK: The Lie algebra $\mathfrak{g}_0' := \mathfrak{so}(b_2 - 3, 3)$ is generated by the Weil maps W_I for all complex structures I of hyperkähler type. The corresponding Lie group G_0' acts as $\operatorname{Spin}(b_2 - 3, 3)$ in odd-dimensional cohomology and $SO(b_2 - 2, 3)$ on even-dimensional ones.

COROLLARY: The natural action of $\mathfrak{g}_0' := \mathfrak{so}(b_2 - 3, 3) = \mathfrak{so}(H^2(M, \mathbb{R}), q)$ on $H^2(M)$ **is extended to an action on the algebra $H^*(M)$ by automorphisms.** Moreover, **this action preserves the Chern classes.**

This finishes the proof of Theorem 1.

Automorphisms of cohomology.

THEOREM: Let M be a simple hyperkähler manifold, and $G\subset GL(H^*(M))$ a group of automorphisms of its cohomology algebra preserving the Pontryagin classes. Then G acts on $H^2(M)$ **preserving the BBF form.** Moreover, the map $G\longrightarrow O(H^2(M,\mathbb{R}),q)$ **is surjective on a connected component, and has compact kernel.**

Proof. Step 1: Fujiki formula $v^{2n}=q(v,v)^n$ implies that Γ_0 **preserves the Bogomolov-Beauville-Fujiki up to a sign.** The sign is fixed, if n is odd.

Step 2: For even n, the sign is also fixed. Indeed, G preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2}\wedge p_1(M)=q(v,v)^{n-1}c$, for some $c\in\mathbb{R}$. The constant c is positive, **because the degree of $c_2(B)$ is positive** for any Yang-Mills bundle with $c_1(B)=0$.

Step 3: $\mathfrak{so}(H^2(M,\mathbb{R}),q)$ acts on $H^*(M,\mathbb{R})$ by derivations preserving Pontryagin classes (Theorem 1). Therefore Lie(G) surjects to $\mathfrak{o}(H^2(M,\mathbb{R}),q)$.

Step 4: The kernel K of the map $G\longrightarrow G\big|_{H^2(M,\mathbb{R})}$ is compact, because it commutes with the Hodge decomposition and Lefschetz $\mathfrak{sl}(2)$-action, hence preserves the Riemann-Hodge form, which is positive definite. ■

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

Proof: (i) is clear from the $\operatorname{Spin}(H^2(M,\mathbb{R}))$-action on $H^*(M)$, and (ii) follows because the kernel K of the map $\operatorname{Aut}(H^*(M)) \longrightarrow G\big|_{H^2(M,\mathbb{R})}$ is compact, hence the discrete group $\Gamma_0 \cap K$ is finite. ■

COROLLARY: The mapping class group of M is mapped to $O(H^2(M,\mathbb{Z}),q)$ with finite kernel and finite index.