Teichmüller spaces for geometric structures, lecture 4

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Hyperkähler manifolds (reminder)

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds (reminder)

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2, 0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Bogomolov-Beauville-Fujiki form (reminder)

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(b_2-3,3)$.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

The Teichmüller space and the mapping class group (reminder)

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\operatorname{Teich}}$ the space of complex structures on M, and let $\operatorname{Teich} := \widetilde{\operatorname{Teich}}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}(M)$ be the group of diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ **the mapping class group**.

REMARK: For hyperkähler manifolds, we take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler type**. It is open in the usual Teichmüller space.

The period map (reminder)

Remark: For any $J \in \text{Teich}$, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $P: \text{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $P: \text{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: P maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

It is called **the period space** of M.

Global Torelli theorem and mapping class group (reminder)

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (D. Huybrechts)
If I_1, $I_2 \in \operatorname{Teich}$ are non-separable points, then $P(I_1) = P(I_2)$, and (M, I_1) **is birationally equivalent** to (M, I_2)

DEFINITION: Let M be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called **a Hausdorff reduction** of M.

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: ("Global Torelli theorem")
The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ **is a diffeomorphism,** for each connected component of Teich_b.

Theorem: ("Mapping class group is arithmetic")
Let M be a simple hyperkähler manifold, and Γ the mapping class group.

Then

(i) $\Gamma\Big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of** $O(H^2(M,\mathbb{Z}),q)$**.**

(ii) The map $\Gamma \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

Ergodic complex structures

DEFINITION: Let (M, μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: (ergodic actions have dense orbits)
Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof. Step 1: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore the set Z_U of such orbits has measure 0.

Step 2: Choose a countable base $\{U_i\}$ of topology on M. Then the set of points in dense orbits is $M \backslash \bigcup_i Z_{U_i}$. ■

DEFINITION: Let M be a complex manifold, Teich its Techmüller space, and Γ the mapping group acting on Teich. **An ergodic complex structure** is a complex structure with dense Γ-orbit.

CLAIM: Let (M, I) be a manifold with ergodic complex structure, and I' another complex structure. **Then there exists a sequence of diffeomorphisms ν_i such that $\nu_i^*(I)$ converges to I'.**

Ergodicity of the monodromy group action

DEFINITION: A lattice in a Lie group is a discrete subgroup $\Gamma \subset G$ such that G/Γ has finite volume with respect to Haar measure.

THEOREM: (Calvin C. Moore, 1966) Let Γ be a lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact subgroup. **Then the left action of Γ on G/H is ergodic.**

THEOREM: Let $\mathbb{P}\mathrm{er}$ be a component of a birational Teichmüller space, and Γ its monodromy group. Let $\mathbb{P}\mathrm{er}_e$ be a set of all points $L \subset \mathbb{P}\mathrm{er}$ such that the orbit $\Gamma \cdot L$ is dense (such points are called **ergodic**). **Then $Z := \mathbb{P}\mathrm{er} \setminus \mathbb{P}\mathrm{er}_e$ has measure 0.**

Proof. Step 1: Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. **Then Γ-action on G/H is ergodic,** by Moore's theorem.

Step 2: Ergodic orbits are dense, non-ergodic orbits have measure 0. ■

REMARK: Generic deformation of M has no rational curves, and no non-trivial birational models. Therefore, **outside of a measure zero subset, $\mathrm{Teich} = \mathrm{Teich}_b$.** This implies that **almost all complex structures on M are ergodic.**

Ratner's theorem

DEFINITION: Let G be a connected Lie group equipped with a Haar measure. **A lattice** $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, G/Γ has finite volume).

EXAMPLE: By Borel and Harish-Chandra theorem, **any integer lattice in a simple Lie group has finite covolume.**

DEFINITION: Unipotent element in a Lie group is an exponent of a nilpotent element.

THEOREM: Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ an arithmetic lattice. Then **the closure of any Γ-orbit in G/H is an orbit of a Lie subgroup $S \subset G$, such that $S \cap \Gamma \subset S$ is a lattice.**

EXAMPLE: Let V be a real vector space with a non-degenerate bilinear symmetric form of signature $(3, k)$, $k > 0$, $G := SO^+(V)$ a connected component of the isometry group, $H \subset G$ a subgroup acting trivially on a given positive 2-dimensional plane, $H \cong SO^+(1, k)$, and $\Gamma \subset G$ an arithmetic lattice. Consider the quotient $\mathbb{P}\mathrm{er} := G/H$. **Then a closure of $\Gamma \cdot J$ in G/H is an orbit of a closed connected Lie subgroup $S \subset G$ containing H.**

Classification of Γ-orbits on $\mathbb{P}\mathrm{er}$

CLAIM: Let $G = SO(3,k)$ be a group of oriented isometries of $V = \mathbb{R}^{3,k}$, and $H \cong SO(1,k) \subset G$. Denote by $\mathfrak{h}$, $\mathfrak{g}$ their ie algebras. Then **any Lie algebra $\mathfrak{s}$ such that $\mathfrak{h} \subsetneq \mathfrak{s} \subsetneq \mathfrak{g}$ is isomorphic to $\mathfrak{so}(2,k)$.** This is the Lie algebra of the Lie group $S = SO(2,k)$ fixing a positive vector $v \in V$.

COROLLARY: Let $J \in \mathbb{P}\mathrm{er} = G/H$, and $\Gamma \subset G$ be am arithmetic lattice. **Then one of three things happens.**

(i) ether J is ergodic,

(ii) or the closure of Γ-orbit of J is an orbit of S or its connected component S^+,

(iii) or the orbit $\Gamma \cdot J$ is closed.

Characterization of ergodic complex structures

REMARK: By Ratner's theorem, the S^+-orbit of J in (ii) and the H-orbit of J in (iii) has finite volume in G/Γ Therefore, **its intersection with Γ is a lattice in H.** This brings

COROLLARY: Consider the action of the mapping class group Γ of a hyperkähler manifold on its period space $\mathbb{P}\mathrm{er}$. Let $J \in \mathbb{P}\mathrm{er}$ be a point such that its Γ-orbit is closed in $\mathbb{P}\mathrm{er}$. Consider its stabilizer $\mathrm{St}(J) \cong H \subset G$. **Then $\mathrm{St}(J) \cap \Gamma$ is a lattice in $\mathrm{St}(J)$.**

COROLLARY: Let J be a complex structure with closed Γ-orbit on a hyperkähler manifold, Ω its holomorphic symplectic form, and $W \subset H^2(M, \mathbb{R})$ a plane generated by $\operatorname{Re}\Omega, \operatorname{Im}\Omega$. **Then W is rational.**

Similarly, one has

COROLLARY: Let J be a non-ergodic complex structure on a hyperkähler manifold, and $W \subset H^2(M, \mathbb{R})$ be a plane generated by $\operatorname{Re}\Omega, \operatorname{Im}\Omega$. **Then W contains a rational vector.**