Teichmüller spaces for geometric structures, lecture 5

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Holomorphically symplectic manifolds (reminder)

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Teichmüller space and the mapping class group (reminder)

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\operatorname{Teich}}$ the space of complex structures on M, and let $\operatorname{Teich} := \widetilde{\operatorname{Teich}}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}(M)$ be the group of diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ **the mapping class group**.

REMARK: For hyperkähler manifolds, we take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler type**. It is open in the usual Teichmüller space.

The period map (reminder)

Remark: For any $J \in \operatorname{Teich}$, (M,J) is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.

Definition: Let $P: \operatorname{Teich} \longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map $P: \operatorname{Teich} \longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

REMARK: P maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid \ q(l,l) = 0, q(l,\bar{l}) > 0\},$$

where $q \in \operatorname{Sym}^2(H^2(M,\mathbb{Z}))$ is the BBF form. It is called **the period space** of M.

Global Torelli theorem and mapping class group (reminder)

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (D. Huybrechts)
If I_1, $I_2 \in \operatorname{Teich}$ are non-separable points, then $P(I_1) = P(I_2)$, and (M, I_1) **is birationally equivalent** to (M, I_2)

DEFINITION: Let M be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called **a Hausdorff reduction** of M.

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: ("Global Torelli theorem")
The period map $\operatorname{Teich}_b \stackrel{\operatorname{Per}}{\longrightarrow} \mathbb{P}\mathrm{er}$ **is a diffeomorphism,** for each connected component of Teich_b.

Theorem: ("Mapping class group is arithmetic")
Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of** $O(H^2(M,\mathbb{Z}), q)$**.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}), q)$ **has finite kernel.**

Ergodic complex structures (reminder)xs

DEFINITION: Let M be a complex manifold, Teich its Techmüller space, and Γ the mapping group acting on Teich. **An ergodic complex structure** is a complex structure with dense Γ-orbit.

CLAIM: Let (M, I) be a manifold with ergodic complex structure, and I' another complex structure. **Then there exists a sequence of diffeomorphisms ν_i such that $\nu_i^*(I)$ converges to I'.**

THEOREM: Let J be a non-ergodic complex structure on a hyperkähler manifold, Ω_J its holomorphic symplectic form, and $W \subset H^2(M, \mathbb{R})$ a plane generated by $\operatorname{Re}\Omega_J, \operatorname{Im}\Omega_J$. **Then W contains a rational vector.**

THEOREM: Let J be a complex structure on a hyperkähler manifold, Ω_J its holomorphic symplectic form, and $W \subset H^2(M, \mathbb{R})$ a plane generated by $\operatorname{Re}\Omega_J, \operatorname{Im}\Omega_J$. If W is rational, the corresponding Γ-orbit is closed. If W is irrational, but contains a rational vector v, then **the closure of the Γ-orbit of J is all $I \in$ Teich such that $\operatorname{Re}\Omega_J, \operatorname{Im}\Omega_J$ contains v.**

Twistor spaces and hyperkähler geometry

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$

DEFINITION: A **twistor space** $\text{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\text{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\text{Tw}} = I_m \oplus I_J : T_x \text{Tw}(M) \to T_x \text{Tw}(M)$ satisfies $I_{\text{Tw}}^2 = -\text{Id}$. **It defines an almost complex structure on $\text{Tw}(M)$.** This almost complex structure is known to be integrable (Obata).

Non-hyperbolic manifolds

DEFINITION: An entire curve in a complex manifold is an image of $\mathbb{C}$ under a non-constant holomorphic map.

REMARK: Let (M, I_k) be a sequence of complex structures on M converging to I. Assume that all (M, I_k) contain an entire curve. **Then (M, I) contains an entire curve.** This result follows from Brody lemma.

DEFINITION: A complex manifold containing no entire curves is called **Kobayashi hyperbolic**. A complex manifold containing an entire curve is called **non-hyperbolic**.

REMARK: By Brody lemma, **non-hyperbolicity is equivalent to degeneracy of Kobayashi metric,** defined later today.

Entire curves in twistor fibers

THEOREM: (F. Campana, 1992)
Let M be a hyperkähler manifold, and $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ its twistor projection. **Then there exists an entire curve in some fiber of π.**

CLAIM: There exists a twistor family which has only ergodic fibers.

Proof: A twistor curve $\mathbb{C}P^1 \subset \mathbb{P}\mathrm{er}$ associated with a 3-plane $W \subset H^2(M,\mathbb{R})$ without rational vectors does not contain any non-ergodic complex structures.
■

THEOREM: All hyperkähler manifolds are non-hyperbolic.

Proof: Let $\operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$ be a twistor family with all fibers ergodic. **By Campana's theorem, one of these fibers, denoted (M,I), is non-hyperbolic.** Since any complex structure $I' \in \mathrm{Teich}$ lies in the closure of $\operatorname{Diff}(M) \cdot I$, all complex structures $I' \in \mathrm{Teich}$ are non-hyperbolic. ■

Kobayashi pseudometric

DEFINITION: A **pseudometric** on a space M is a function $\operatorname{Sym}^2(M) \longrightarrow \mathbb{R}^{\geqslant 0}$ satisfying the triangle inequality (almost like a metric, but can vanish anywhere).

REMARK: Supremum of a family of pseudometrics is again a pseudometric.

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M is the supremum of all pseudometric on M such that any holomorphic map from the Poincaré disk to M is distance-decreasing.

THEOREM: Let $\pi: \mathcal{M} \longrightarrow X$ be a smooth holomorphic family, which is trivialized as a smooth manifold: $\mathcal{M} = M \times X$, and d_x the Kobayashi metric on $\pi^{-1}(x)$. **Then $d_x(m, m')$ is upper continuous on x.** ■

COROLLARY: Denote the diameter of the Kobayashi pseudometric by $\operatorname{diam}(d_x) := \sup_{m,m'} d_x(m, m')$. **Then $\operatorname{diam}: X \longrightarrow \mathbb{R}^{\geqslant 0}$ is upper continuous** in families of complex manifolds.

Vanishing of Kobayashi pseudometric and Γ-action

In the sequel, **all results are obtained in colaboration with Ljudmila Kamenova and Steven Lu.**

THEOREM: Let (M, I) be a complex manifold with vanishing Kobayashi pseudometric. Then **the Kobayashi pseudometric vanishes for all complex structures J such that I is contained in the Γ-orbit of J.**

Proof: Let $\operatorname{diam}: \operatorname{Comp} \longrightarrow \mathbb{R}^{\geqslant 0}$ map a complex structure J to the diameter of the Kobayashi pseudometric on (M, J). Let J be an ergodic complex structure. The set of points $J' = \nu(J) \in \operatorname{Comp}$, $\nu \in \operatorname{Diff}$, is dense, because J is ergodic. By upper semi-continuity, $0 = \operatorname{diam}(I) \geqslant \inf_{J'=\nu(J)} \operatorname{diam}(J') = \operatorname{diam}(J)$.
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Vanishing of Kobayashi pseudometric: examples

EXAMPLE: Let M be a projective K3 surface. Then the Kobayashi metric on M vanishes. **Since all non-projective K3 are obtained as limits of projective ones,** the Kobayashi metric vanishes on non-projective K3 surfaces as well.

REMARK: Let M be a hyperkähler manifold. **"Hyperkähler SYZ conjecture"** states that a deformation of M admits a holomorphic Lagrangian fibration. It is known for all known examples of hyperkähler manifolds.

THEOREM: Let M be a compact simple hyperkähler manifold. Assume that a deformation of M admits a holomorphic Lagrangian fibration and the Picard rank of M is not maximal. **Then the Kobayashi pseudometric on M vanishes.**

THEOREM: Let M be a Hilbert scheme of K3. **Then the Kobayashi pseudometric on M vanishes.**