Apollonian gaskets and limit set of automorphisms of a K3 surface

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Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

Kähler manifolds

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M. The set of all Kähler classes is called **the Kähler cone**.

REMARK: **(the Hodge decomposition)**
The second cohomology of a compact Kähler manifold are decomposed as $H^2(M, \mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$, where $H^{2,0}(M)$ is the space of all cohomology classes which can be represented by holomorphic $(2,0)$-forms, $H^{0,2}(M)$ its complex conjugate, and $H^{1,1}(M)$ the classes which can be represented by I-invariant forms.

Kummer surfaces

REMARK: Everything I will be talking about today works not only for K3, but for hyperkähler manifolds of maximal holonomy. **I will present it for K3 to save the time and effort.**

DEFINITION: A **holomorphically symplectic manifold** is a complex manifold equipped with a non-degenerate, holomorphic (2,0)-form.

EXAMPLE: For any complex manifold M, **the total space T^*M of the cotangent bundle is holomorphically symplectic.**

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity $\mathbb{C}^2/\pm 1$.**

REMARK: Let M be a 2-dimensional complex manifold which is holomorphic symplectic form outside of singularities, which are all of form $\mathbb{C}^2/\pm 1$. Then **its resolution is also holomorphically symplectic.**

DEFINITION: Take a 2-dimensional complex torus T, then all 16 singular points of $T/\pm 1$ are of this form. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

DEFINITION: A K3 surface is a complex deformation of a Kummer surface.

K3 surfaces

"K3: Kummer, Kähler, Kodaira" (the name is due to A. Weil).

"Faichan Kangri is the 12th highest mountain on Earth."

Topology of K3 surfaces

THEOREM: Any complex compact surface with $c_1(M) = 0$ and $H^1(M) = 0$ **is isomorphic to K3.** Moreover, **it is Kähler.**

CLAIM: 1. $\pi_1(K3) = 0$,

2. The second homology and cohomology of K3 is torsion-free.

3. $b_2(K3) = 22$, and the signature of its intersection form is $(3, 19)$.

4. The intersection form of K3 is even, and the corresponding quadratic lattice is $U^3 \oplus (-E_8)^2$, where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8 is the Coxeter matrix for the group E_8.

Complex surfaces and hyperbolic lattices

REMARK: Let M be a complex surface of Kähler type. **Then the signature of the intersection form on $H^{1,1}(M)$ is $(1, h^{1,1} - 1)$.**

THEOREM: Let M be a projective K3 surface, and $\operatorname{Aut}(M)$ its group of complex automorphisms. **Then the natural map $\operatorname{Aut}(M) \longrightarrow O(H^{1,1}(M))$ has finite kernel.**

REMARK: Since $H^{1,1}(M)$ has signature $(1, h^{1,1}-1)$, the group $PSO(H^{1,1}(M))$ is the group of isometries of a hyperbolic space $\mathbb{P}H^{1,1}(M)$. If we are interested in dynamics, the "finite kernel" does not make any difference. **The automorphisms of M can be classified in the same way as isometries of the hyperbolic space of constant sectional curvature.**

Classification of automorphisms of a hyperbolic space

REMARK: The group $O(m,n), m,n > 0$ has 4 connected components. We denote the connected component of 1 by $SO^+(m,n)$. We call a vector v **positive** if its square is positive.

DEFINITION: Let V be a vector space with quadratic form q of signature $(1,n)$, $\operatorname{Pos}(V) = \{x \in V \mid q(x,x) > 0\}$ its **positive cone**, and $\mathbb{P}^+V$ projectivization of $\operatorname{Pos}(V)$. Denote by g any $SO(V)$-invariant Riemannian structure on $\mathbb{P}^+V$. Then $(\mathbb{P}^+V, g)$ is called **hyperbolic space**, and the group $SO^+(V)$ **the group of oriented hyperbolic isometries**.

Theorem-definition: Let $n > 0$, and $\alpha \in SO^+(1,n)$ is an isometry acting on V. Then one and only one of these three cases occurs

(i) α has an eigenvector x with $q(x,x) > 0$ (α is **"elliptic isometry"**)

(ii) α has an eigenvector x with $q(x,x) = 0$ and eigenvalue λ_x satisfying $|\lambda_x| > 1$ (α is **"hyperbolic (or loxodromic) isometry"**)

(iii) α has a unique eigenvector x with $q(x,x) = 0$ and eigenvalue 1. (α is **"parabolic isometry"**)

DEFINITION: An automorphism of a K3 surface (M,I) is called **elliptic (parabolic, hyperbolic)** if it is elliptic (parabolic, hyperbolic) on $H^{1,1}_I(M,\mathbb{R})$.

Shape of the Kähler cone

DEFINITION: Let (M, I) be a compact complex manifold admitting a Kähler structure. Recall that its **Kähler cone** $\operatorname{Kah}(M)$ is the set of all $x \in H^{1,1}(M, I)$ represented by Kähler forms.

REMARK: The Kähler cone of a complex manifold **is open and convex in $H^{1,1}(M, I)$.**

DEFINITION: Let M be a K3 surface. An integer (1,1)-class $\eta \in H^{1,1}(M, \mathbb{Z})$ is called **a (-2)-class** if $\eta^2 = -2$.

PROPOSITION: Let $\eta \in H^{1,1}(M, \mathbb{Z})$ be a (-2)-class on a K3 surface. **Then η or $-\eta$ is represented as the fundamental class of a complex curve**.

THEOREM: Let M be a K3 surface, and $\mathfrak{S} \subset H^{1,1}(M, \mathbb{Z})$ the set of all (-2)-classes represented by a complex curve. **Then $\operatorname{Kah}(M)$ is the set of all $\eta \in H^{1,1}(M, \mathbb{R})$ such that $\eta^2 > 0$ and $\langle \eta, S \rangle > 0$ for all $S \in \mathfrak{S}$.**

(-2)-reflections and Weyl chambers

DEFINITION: Let $\eta \in H^2(M,\mathbb{Z})$ be an integer class on a complex surface with $\eta^2 = -2$. Consider a map $r_\eta : H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathbb{Z})$ taking v to $v + q(v,\eta)\eta$, where q denotes the intersection form. Clearly, r_η acts trivially on $\eta^\perp$ and takes η to $-\eta$. We call this map **the reflection associated with** η.

DEFINITION: Let ω be a Kähler form on a complex surface. Clearly, $\int_M \omega^2 > 0$. Since the intersection form on $H^{1,1}(M,\mathbb{R})$ has signature $(1,k)$, the set of vectors with positive square has 2 connected components. Let **the positive cone** $\operatorname{Pos}(M) \subset H^{1,1}(M,\mathbb{R})$ be a connected component of this set containing the Kähler cone.

DEFINITION: Let M be a K3 surface, and $\mathfrak{R} \subset H^{1,1}(M,\mathbb{Z})$ the set of all (-2)-classes. The **Weyl chamber** is a connected component in the complement $\operatorname{Pos}(M) \backslash \bigcup_{\eta\in\mathfrak{R}} \eta^\perp$.

CLAIM: Let $H \subset O(H^2(M,\mathbb{Z}))$ be the group generated by reflections r_η, for all $\eta \in \mathfrak{R}$. **Then H acts transitively on the set of all Weyl chambers.** ■

The mapping class group

REMARK: Recall that the orthogonal group $O(3,19)$ has 4 connected components. Denote by $O^+(3,19)$ its index 2 subgroup containing the (-2)-reflections. Donaldson proved that **any diffeomorphism of a K3 surface acts on $H^2(M,\mathbb{Z})$ as an element of $O^+(H^2(M,\mathbb{Z}))$.**

CLAIM: The group $O(H^2(M,\mathbb{Z}))$ **is generated by all reflections r_η,** for all (-2)-classes η. Moreover, **for each η there exists a diffeomorphism of M which acts as r_η on $H^2(M,\mathbb{Z})$.**

REMARK: This implies that **the mapping class group (MCG) of K3 acts on $H^2(M,\mathbb{Z})$ as $O^+(H^2(M,\mathbb{Z}))$.**

REMARK: Since the MCG acts transitively on the set of Weyl chambers, **each Weyl chamber serves as a Kähler cone for an appropriate K3**.

Arithmetic lattices

DEFINITION: Let G be a semisimple algebraic Lie group defined over $\mathbb{Q}$. An **arithmetic lattice** is a group $\Lambda \subset G$ which is commensurable with $G_\mathbb{Z}$.

THEOREM: (Borel and Harish-Chandra)
Let G be an algebraic Lie group over $\mathbb{Q}$ which does not have non-trivial rational characters, such as a semisimple group. **Then $\frac{G}{G_\mathbb{Z}}$ has finite Haar measure.**

DEFINITION: **Covolume** of a discrete group Γ acting on a space (M, μ) with measure is $\int_{M/\Gamma} \mu$.

DEFINITION: Recall that the group of isometries of the hyperbolic space $\mathbb{H}^m = \frac{SO^+(1,n)}{SO(n)}$ of constant sectional curvature is $O^+(1,n)$. Let $\Gamma \subset O^+(1,n)$ be a discrete group of finite covolume. **A hyperbolic orbifold** is a quotient $\mathbb{H}^m/\Gamma$. It is called **a hyperbolic manifold** if Γ has no elements of finite order.

REMARK: By Selberg lemma, **any arithmetic group $G_\mathbb{Z} \subset G$ has a finite index subgroup without elements of finite order.**

COROLLARY: Let $\Gamma \subset SO^+(1,n)$ be an arthmetic lattice. **Then $\mathbb{H}^m/\Gamma$ is aa hyperbolic orbifold.**

The automorphism group and Hodge monodromy

DEFINITION: Let (M,I) be a K3 surface. **The group of Hodge monodromy** is the group Mon_I of all $s \in O^+(H^2(M,\mathbb{Z}))$ which preserve the Hodge decomposition $H^2(M,\mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$

DEFINITION: The **Kähler chamber of a K3 surface** is its Kähler cone considered as one of its Weyl chambers.

THEOREM: Let M be a K3 surface. Then the automorphism group of M is discrete, **the restriction map** $\operatorname{Aut}(M) \longrightarrow \operatorname{Aut}(M)\Big|_{H^2(M,\mathbb{Z})}$ **is injective,** and **its image of all** $\tau \in \operatorname{Mon}_I$ **which fix the Kähler chamber.**

The automorphism group of K3 **can be interpreted as the fundamental group of a certain hyperbolic manifold with a boundary.**

The ample cone

DEFINITION: Let $H^{1,1}(M,\mathbb{Q}) := H^2(M,\mathbb{Q}) \cap H^{1,1}(M)$. Define **the positive rational cone** as $\operatorname{Pos}(M) \cap H^{1,1}(M,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$, and **the ample cone** $\operatorname{Kah}_{\mathbb{Q}}(M)$ as the intersection of $\operatorname{Kah}(M)$ and $\operatorname{Pos}_{\mathbb{Q}}(M)$.

REMARK: The group of Hodge monodromy acts on $H^{1,1}(M,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$ as an arithmetic lattice, and **the quotient $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M)/\operatorname{Mon}_I$ is a hyperbolic manifold.**

CLAIM: The group of Hodge monodromy **acts on the set if all (-2)-classes of type (1,1) with finitely many orbits.**

Proof: The image of Mon_I has finite index in $O(H^{1,1}(M,\mathbb{Z}))$. In any non-degenerate quadratic lattice Λ, the group $O(\Lambda)$ acts on the set $V_n := \{v \in \Lambda \ | \ q(v,v) = n\}$ with finite number of orbits, for any given $n \in \mathbb{Z}$. ■

Automorphism group as the fundamental group

DEFINITION: Let $\mathcal{H} := \mathbb{P}\operatorname{Pos}(V)/\Gamma$ be a hyperbolic orbifold, and $S_1, ..., S_n \subset \mathcal{H}$ a finite collection of immersed hyperbolic hypersurfaces of finite volume, associated with hyperspaces $V_i \subset V$ of codimension 1. Any connected component of $\mathcal{H} \backslash \bigcup S_i$ is called **is a convex hyperbolic orbifold with polyhedral boundary**.

COROLLARY: The image of a Weyl chamber in $\mathcal{H} := \mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M)/\operatorname{Mon}_I$ **is a convex hyperbolic orbifold with polyhedral boundary,** obtained as a connected component of $\mathcal{H} \backslash \bigcup_{s \in O} \mathcal{H}_s$, where O is the set of all orbits of Mon_I acting on (-2)-classes of type (1,1), and H_s is the hyperbolic hypersurface $\frac{\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M) \cap s^\perp}{\operatorname{St}_s(\operatorname{Mon}_I)}$. ■

REMARK: Since the ample cone $\operatorname{Kah}_{\mathbb{Q}}(M)$ is convex, the orbifold fundamental group of its image in $\mathcal{H}$ is the subgroup of Mon_I fixing $\operatorname{Kah}_{\mathbb{Q}}(M)$. **This group is equal to the fundamental group of the corresponding hyperbolic orbifold with a boundary.**

COROLLARY: The group of holomorphic automorphisms of a K3 surface is $\pi_1(\mathcal{K})$, where $\mathcal{K}$ is a hyperbolic orbifold with polyhedral boundary obtained as the image of $\mathbb{P}\operatorname{Kah}_{\mathbb{Q}}(M)$ in $\mathcal{H} := \mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M)/\operatorname{Mon}_I$.

The absolute

DEFINITION: Let V be a real space equipped with a scalar product q of signature $(1,k)$, and V_+ the set of all vectors with positive square Then $\mathbb{H}^n = \mathbb{P}V_+$ is the hyperbolic space of constant negative curvature. **The absolute** Abs is the projectivization of the set of all vectors with square 0; it is identified with the boundary of $\mathbb{H}^n$. Clearly, Abs **is diffeomorphic to the sphere S^{n-1}.** Clearly, the union $\mathbb{H}^n \cup \text{Abs}$ is compact.

REMARK: Let $W \subset V$ be a 2-dimensional subspace of signature (1,1). **Then $\mathbb{P}(W \cap \text{Pos}(V)) \subset \mathbb{H}$ is a geodesic, and all geodesics are obtained this way.**

DEFINITION: The endpoints of a geodesic $\gamma \in \mathbb{P}(W \cap \text{Pos}(V))$ are two points $\text{Abs} \cap \mathbb{P}(W)$.

REMARK: Clearly, **any geodesic is uniquely determined by its two endpoints.**

Claim 1: Let γ_1, γ_2 be two geodesics on a hyperbolic space, and $\delta: \gamma_1 \longrightarrow \mathbb{R}$ the distance from a point of γ_1 to γ_2. Let ∞_+, ∞_- be the endpoints of γ_1, considered as points in Abs. **Then $\lim_{x \longrightarrow \infty_+} \delta(x) = \infty$ if γ_2 is not an endpoint of γ_2, and $\lim_{x \longrightarrow \infty_+} \delta(x) < \infty$ otherwise.**

Proof: Left as an exercise. ■

The limit set

CLAIM: Let $\Gamma \subset SO(1,n)$ be a group of hyperbolic isometries acting on $\mathbb{H}^n$, and $a, b \in \mathbb{H}^n$ any two points. Let $\Lambda_a, \Lambda_b \subset$ Abs be the set of all points in Abs obtained as accumulation points for $\Gamma \cdot a, \Gamma \cdot b$. **Then $\Lambda_a = \Lambda_b$.**

Proof: Let $x \in \Lambda_a$, and $\{y_i = \gamma_i a\} \in \Gamma \cdot a$ be a sequence converging to x. Since $d(\gamma_i b, \gamma_i a) < \infty$, the sequence converges to x as well. Indeed, since $\mathbb{H}^n \cup$ Abs is compact, otherwise we would have a point $y \neq x$ which is a limit of a subsequence $\gamma_i b$. However, **for any two sequences of points $\{x_i\}$ converging to x and $\{y_i\}$ converging to y, we have $\lim_i d(x_i, y_i) = \infty$,** giving a contradiction. ■

DEFINITION: In these assumptions, **the limit set** of Γ is Λ_a, for $a \in \mathbb{H}$.

REMARK: Γ acts on Abs by conformal equivalences, and it acts on Abs$\backslash\Lambda$ properly discontinuously.

Quasi-Fuchsian groups

DEFINITION: A Kleinian group is a discrete subgroup of $PSL(2,\mathbb{C}) = SO^+(1,3)$.

EXAMPLE: Let $\Gamma \subset PSL(2,\mathbb{R})$ be a fundamental group of a Riemann surface $C = \mathbb{H}^2/\Gamma$. We embed $PSL(2,\mathbb{R})$ to $PSL(2,\mathbb{C})$. This gives an embedding of $\Gamma = \pi_1(C)$ to $SO^+(1,3)$; its image is called **a Fuchsian group**.

REMARK: Clearly, the limit set of Γ is the circle $\operatorname{Abs}(\mathbb{R}^{1,2}) \subset \operatorname{Abs}(\mathbb{R}^{1,3}) = S^2$. The group Γ acts on two half-spheres S^+, S^- conformally, inducing a conformal (that is, complex) structure on manifolds S^+/Γ and S^-/Γ.

CLAIM: The Riemannian surface S^+/Γ is isomorphic to C, and S^-/Γ to its complex conjugate $\overline{C}$.

Proof: Left as an exercise. ■

DEFINITION: A Kleinian group $\Gamma \subset SO^+(1,3)$ is called **quasi-Fuchsian** if its limit set $\Lambda \subset S^2$ is a Jordan curve.

REMARK: If this Jordan curve is real analytic somewhere, it is Fuchsian.

Ahlfors double uniformization theorem

THEOREM: A small deformation of a quasi-Fuchsian representation $\Gamma \longrightarrow SO^+(1,3)$ **is always quasi-Fuchsian.**

THEOREM: (Ahlfors double uniformization theorem) Let $\Gamma \subset SO^+(1,3)$ be a quasi-Fuchsian subgroup, and $\Lambda \subset \mathbb{C}P^1$ its limit set, splitting $\mathbb{C}P^1$ onto two half-spheres S^+, $S^- \subset \mathbb{C}P^1$. Consider the map $\Psi: F \longrightarrow \text{Teich} \times \text{Teich}$ from the set F of all quasi-Fuchsian subgroup deformations of Γ to the point of $\text{Teich} \times \text{Teich}$ represented by $(S^+/\Gamma, S^-/\Gamma)$. **Then Ψ is bijective.**

"a hyperbolic hypersurspace"

REMARK: Consider a hyperbolic space $\mathbb{H}^n = \mathbb{P}\operatorname{Pos}(V)$, where V is a real vector space equipped with a scalar product of signature $(1, n)$. For any subspace $W \subset V$ of signature $(1, n-1)$, **the space $\mathbb{P}\operatorname{Pos}(W) \subset \mathbb{P}\operatorname{Pos}(V)$ is a completely geodesical hyperbolic hypersurface, and all completely geodesical hyperbolic hypersurfaces $\mathbb{H}^{n-1} \subset \mathbb{H}^n$ are obtained this way.**

DEFINITION: Later on, when we say **"a hyperbolic hypersurspace",** we always mean $\mathbb{P}\operatorname{Pos}(W) \subset \mathbb{P}\operatorname{Pos}(V)$ as above.

Constructing convex hyperbolic orbifold with polyhedral boundary

REMARK: Let $\Gamma \subset SO^+(1,n)$ be a group of isometries of finite covolume (that is, a lattice), and $\mathbb{P}\operatorname{Pos}(W) \subset \mathbb{P}\operatorname{Pos}(V)$ a hyperbolic hypersurspace. Clearly, **the image of $\mathbb{P}\operatorname{Pos}(W)$ in $\mathbb{P}\operatorname{Pos}(V)/\Gamma$ has the same volume as $\mathbb{P}\operatorname{Pos}(W)/\Gamma_W$**, where Γ_W is $\{\gamma \in \Gamma \ \mid \ \gamma(W) = W\}$. Any integer lattice in $O(1,n)$ has finite covolume. This implies the following description of a convex hyperbolic orbifold with polyhedral boundary.

CLAIM: Let $V_{\mathbb{Z}}$ be a lattice equipped with a scalar product of signature $(1,n)$, and $\Gamma \subset SO^+(V)$ a subgroup commensurable with $SO(V_{\mathbb{Z}})$. Consider a finite collection of rational hyperspaces $W_i \subset V$ of signature $(1, n-1)$, let $\mathfrak{S}$ be $\bigcup_i \Gamma W_i$, and P a connected component of $\mathbb{P}\operatorname{Pos} V \backslash \mathbb{P}\mathfrak{S}$. Denote by Γ_P the group $\{\gamma \in \Gamma \ \mid \ \gamma(P) = P\}$. **Then P/Γ_P is a convex hyperbolic orbifold with polyhedral boundary, and all convex hyperbolic orbifolds with polyhedral boundary in $\mathbb{P}\operatorname{Pos} V/\Gamma$ are obtained this way.**

Today we are interested in the following question.

QUESTION: What is the limit set of Γ_P acting on P?

REMARK: The following theorem is true for all hyperbolic manifolds, but it is much easier to state and prove when $\mathcal{H}$ is compact. **Later I will explain how to generalize it to all hyperbolic $\mathcal{H}$.**

The boundary of a polyhedron

THEOREM: Let $\mathcal{H} := \mathbb{P}\operatorname{Pos} V/\Gamma$ be a hyperbolic manifold, and $\mathcal{P} \subset \mathcal{H}$ a convex hyperbolic orbifold with polyhedral boundary. Let P be a connected component of the preimage of $\mathcal{P}$ in $\mathbb{P}\operatorname{Pos} V = \mathbb{H}$; clearly, P is a convex polyhedron in $\mathbb{H}$ with hyperbolic faces. Denote by $\operatorname{Abs} P$ the set of all points on Abs obtained as limits of $x_i \in P$. **Assume that $\mathcal{H}$ is compact. Then $\operatorname{Abs} P$ is the limit set of Γ_P acting on P.**

Proof. Step 1: Clearly, for any $x \in P$, its orbit belongs to P, **hence its limit set belongs to $\operatorname{Abs} P$.**

Step 2: Conversely, let $x \in \operatorname{Abs} P$, and let γ be a geodesic with an end in x. Assume that γ contains a point in P. Since P is convex, a ray $\gamma_+ \subset \gamma$ converging to x also belongs to P. Choose a fundamental domain D of Γ_P-action on P. Since $\overline{P}/\Gamma_P$ is a closed subset of $\mathcal{H}$, this space is compact, hence D can be chosen compact. Let $R := \operatorname{diam} D$.

Step 3: Choose $z \in D$. Since $\operatorname{diam} D = R$, an R-neighbourhood of any point $y \in \gamma_+$ contains γz for some $\gamma \in \Gamma_P$. Choose a sequence of points y_i converging to $x \in \operatorname{Abs} P$, and let $\gamma_i z$ be points which satisfy $d(\gamma_i z, y_i) \leqslant R$. Then $\lim \gamma_i z$ is a point in an R-neighbourhood $\gamma_+(R)$ of γ_+. By Claim 1, **any unbounded sequence in $\gamma_+(R)$ converges to x, hence x belongs to the limit set of Γ_P.** ∎

Cusp points

DEFINITION: A horosphere on a hyperbolic space is a sphere which is everywhere orthogonal to a pencil of geodesics passing through one point at infinity, and **a horoball** is a ball bounded by a horosphere. **A cusp point** for an n-dimensional hyperbolic manifold $\mathbb{H}/\Gamma$ is a point on the boundary $\partial\mathbb{H}$ such that its stabilizer in Γ contains a free abelian group of rank $n-1$. Such subgroups are called **maximal parabolic**.

CLAIM: For any point $p \in \partial\mathbb{H}$ stabilized by $\Gamma_0 \subset \Gamma$, and any horosphere S tangent to the boundary in p, Γ_0 acts on S by isometries. In such a situation, **p is a cusp point if and only if $(S\backslash p)/\Gamma_0$ is compact.**

REMARK: A cusp point p yields a **cusp** in the quotient $\mathbb{H}/\Gamma$, that is, a geometric end of $\mathbb{H}/\Gamma$ of the form $B/\mathbb{Z}^{n-1}$, where $B \subset \mathbb{H}$ is a horoball tangent to the boundary at p.

Cusp points and thick-thin decomposition

THEOREM: (Thick-thin decomposition) Any n-dimensional complete hyperbolic manifold of finite volume **can be represented as a union of a "thick part", which is a compact manifold with a boundary, and a "thin part", which is a finite union of quotients of form $B/\mathbb{Z}^{n-1}$, where B is a horoball tangent to the boundary at a cusp point, and $\mathbb{Z}^{n-1} = \mathrm{St}_\Gamma(B)$.**

THEOREM: (A. Borel) Let $V_\mathbb{Z}$ be a lattice equipped with a scalar product of signature $(1, n)$, and $\Gamma \subset SO^+(V)$ a subgroup commensurable with $SO(V_\mathbb{Z})$. Then the cusp points of the hyperbolic manifold $\mathbb{H}/\Gamma$ are in bijective correspondence with Γ-orbits on the set $\mathrm{Abs} \cap \mathbb{P}(V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q})$ of all rational points on Abs.

The boundary of a polyhedron,

THEOREM: Let $\mathcal{H} := \mathbb{P}\operatorname{Pos} V/\Gamma$ be a hyperbolic manifold, and $\mathcal{P} \subset \mathcal{H}$ a convex hyperbolic orbifold with polyhedral boundary. Let P be a connected component of the preimage of $\mathcal{P}$ in $\mathbb{P}\operatorname{Pos} V = \mathbb{H}$; clearly, P is a convex polyhedron in $\mathbb{H}$ with hyperbolic faces. Denote by $\operatorname{Abs} P$ the set of all points on Abs obtained as limits of $x_i \in P$ **Then $\operatorname{Abs} P$ is the union of all cusp points in $\operatorname{Abs} P$ and the limit set of Γ_P acting on P.**
Proof. Step 1: Clearly, for any $x \in P$, its Γ_P orbit belongs to P, **hence its limit set belongs to $\operatorname{Abs} P$.**

Step 2: Consider the thick-thin decomposition of $\mathcal{H}$, with the thick part $\mathcal{H}_0$, and let P_0 be the intersection of P and the preimage of $\mathcal{H}_0$. Then P_0 is obtained by removing from P a countable number of horoballs; this set is Γ_P-invariant. **The fundamental domain D of Γ_P-action on P_0 has finite diameter R.**

Step 3: Fix a non-cusp point $x \in \operatorname{Abs} P$, and fix $z \in P$. We need to show that $\gamma_i z$ converges to x for some sequence $\{\gamma_i\} \in \Gamma_P$. Let $\gamma_+ \subset P$ be a geodesic ray with an end in x. An intersection of a horoball and a geodesic ray is bounded, unless its end coincides with the boundary point of the horoball. Then γ_+ contains a family of points $\{y_i\} \subset \gamma_+ \cap P_0$ converging to x. **This implies that an R-neighbourhood of y_i contains $\gamma_i D$, hence it contains $\gamma_i z$, and $\lim \gamma_i z = x$ by another application of Claim 1.** ■

The Apollonian gasket

DEFINITION: Let $V_{\mathbb{Z}}$ be a lattice equipped with a scalar product of signature $(1, n)$, and $\Gamma \subset SO^+(V)$ a subgroup commensurable with $SO(V_{\mathbb{Z}})$. Consider a convex hyperbolic orbifold $\mathcal{P} \subset \mathcal{H}$ with polyhedral boundary. Let P be a connected component of the preimage of $\mathcal{P}$ in $\mathbb{P}\operatorname{Pos} V = \mathbb{H}$, and $\operatorname{Abs} P$ the set of accumulation points of P on Abs. **The Apollonian gasket** is the union of all positive-dimensional real analytic subvarieties in $\operatorname{Abs} P$.

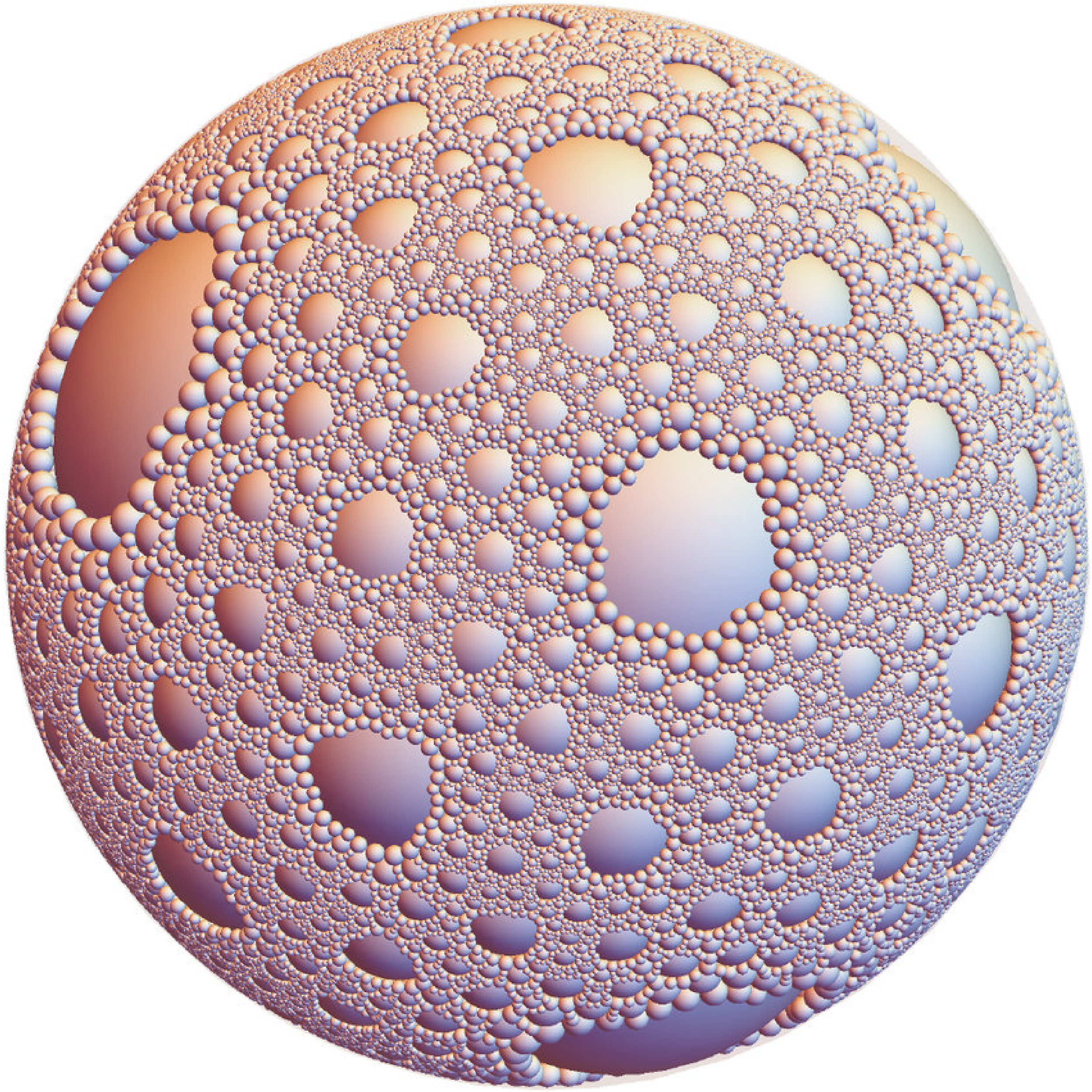
THEOREM: The Apollonian gasket of P **is a union of spheres** $\operatorname{Abs} \cap \mathbb{P}W_i$**, where** $W_i \subset V$ **is a rational subspace of signature** $(1, k)$**.**

The seven mother goddesses

The Seven Mother Goddesses (Matrikas) Flanked by Shiva-Virabhadra and Ganesha, Lord of Obstacles, India, Madhya Pradesh, 9th century

REMARK: In a sequence of papers, A. Baragar **found many examples of K3 surfaces with Apollonian gasket on the boundary.**

Apollonian sphere packings



Pentatope-based Apollonian packing, by Michael Fennen and Domenico Giulini, "Lie sphere geometry in lattice cosmology", 2020

Ergodic group action

DEFINITION: Let (M, μ) be a space with finite measure, and G a group acting on M preserving μ. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof. Step 1: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore the set Z_U of such orbits has measure 0.

Proof. Step 2: Choose a countable base $\{U_i\}$ of topology on M. Then the set of points in dense orbits is $M \backslash \bigcup_i Z_{U_i}$. ■

CLAIM: A group G acts on M ergodically **if and only if any L^2-integrable G-invariant function on M is constant almost everywhere.**

Shah's theorem

DEFINITION: Let $ST\mathbb{H}^n$ be the manifold of unit tangent vectors to a hyperbolic space of constant sectional curvature, and Vis : $ST\mathbb{H}^n \longrightarrow$ Abs a map taking a tangent vector to the limit point of the corresponding geodesic.

DEFINITION: Let $(v, m) \in ST\mathbb{H}^n$, with $v \in T_m\mathbb{H}^n$ being a unit tangent vector. Denote by $\gamma_{v,m}(t)$ the geodesic starting to m and tangent to v. **Geodesic flow** is a measure-preserving flow of diffeomorphisms $G_t : ST\mathbb{H}^n \times \mathbb{R} \longrightarrow ST\mathbb{H}^n$ taking $(v, m), t$ to the tangent vector $\gamma'_{v,m}(t)$.

REMARK: By Hopf theorem, the geodesic flow on a hyperbolic manifold is ergodic. However (unlike for Ratner's theorem on homogeneous flows), **the closure of a geodesic might be very pathological,** such as a product of a Cantor set and an interval. Shah's theorem can be used to rectify this problem.

THEOREM: (Nimish Shah, "LIMITING DISTRIBUTIONS OF CURVES UNDER GEODESIC FLOW ON HYPERBOLIC MANIFOLDS", 2007, Theorem 1.2) Let $\mathcal{H} = \mathbb{H}^n/\Gamma$ be a hyperbolic orbifold, and $C \subset STM$ a real analytic interval, such that Vis(C) is not a singleton. Denote by X the minimal closed subset of STM invariant under the geodesic flow and containing C. **Then** Vis(X) **is a totally geodesic sphere in** Abs**.** Moreover, **the image of its convex hull is a hyperbolic submanifold in $\mathcal{H}$.**

Shah's theorem and hyperbolic polyhedra

THEOREM: The Apollonian gasket of P **is a union of spheres** $\operatorname{Abs}\cap\mathbb{P}W_i$**, where** $W_i \subset V$ **is a rational subspace of signature** $(1,k)$**.**

Proof. Step 1: Let $C \subset \operatorname{Abs} P$ be a real analytic curve. Connecting a point in P with C by a real analytic family of geodesics, **we obtain a real analytic path** $C_1 \subset P$ **satisfying assumptions of Shah's theorem.** Running a geodesic flow on its closure, we obtain a geodesic sphere S on Abs which belongs to $\operatorname{Abs} P$ and contains C.

Step 2: The convex hull of $S = \mathbb{P}W \cap \operatorname{Abs}$ is the hyperbolic subspace $\mathbb{P}\operatorname{Pos} W$. By Ratner theorem, its image in $\mathcal{H} = \mathbb{P}\operatorname{Pos} V/\Gamma$ is closed if and only if $\Gamma_W := \{\gamma \in \Gamma \ \mid\ \gamma(W) = W\}$ is a lattice in $O(W)$, which happens if and inly if W is rational. ■

Apollonian gasket for a K3 surface

DEFINITION: Let S be a sphere which lies in the Apollonian gasket. We say that S **is a component of the Apollonian gasket** if S does not belong to a sphere of bigger dimension, which also lies in the Apollonian gasket. Note that **the components can be tangent or intersect each other.**

DEFINITION: **The Apollonian gasket of a K3 surface** is the union of all geodesic spheres which belong to the boundary of its ample cone.

REMARK: Clearly, **the Apollonian gasket of a K3 surface is the Apollonian gasket of the convex polyhedron with hyperbolic faces obtained as the image of its ample cone.**

Apollonian gasket for a K3 surface: explicit description

THEOREM: Let M be a projective K3 surface, $NS := H^{1,1}(M,\mathbb{R}) \cap H^2(M,\mathbb{Z})$ its Neron-Severi lattice. Consider the set of all rational subspaces $W_i \subset NS_{\mathbb{R}}$ of signature $(1,k)$, $k \geqslant 2$ such that for any (-2)-class $\eta \in NS$, with $\eta \not\perp W_i$, and any integer class $\rho \in W_i^\perp$, the space $\langle \rho, \eta \rangle$ is not negative definite. **Then the sphere S_i Abs $\cap \mathbb{P}W_i$ belongs to one of the Weyl chambers of M**, associated with the complex structure I_i on M. Moreover, S_i belongs to the Apollonian gasket of (M, I_i), and **all components of the Apollonian gasket of M are obtained this way.**

REMARK: This gives an effective and easy way to determine the Apollonian gasket for special examples when we have good control over the lattice; by Nikulin's theorem, **any even quadratic lattice of signature $(1,n)$, $n \leqslant 10$ can be realized as $NS(M,I)$ for an appropriate K3 surface.**

Apollonian gasket for a K3 surface: explicit description (2)

THEOREM: Let M be a projective K3 surface, $NS := H^{1,1}(M,\mathbb{R})\cap H^2(M,\mathbb{Z})$ its Neron-Severi lattice. Consider the set of all rational subspaces $W_i \subset NS_{\mathbb{R}}$ of signature $(1,k)$, $k \geqslant 2$ such that for any (-2)-class $\eta \in NS$, not orthogonal to W_i, the space $\eta + W_i^\perp$ is not negative definite. **Then the sphere $S_i :=$ Abs$\cap\mathbb{P}W_i$ belongs to one of the Weyl chambers of M**, associated with the complex structure I_i on M. Moreover, S_i belongs to the Apollonian gasket of (M,I_i), and **all components of the Apollonian gasket of M are obtained this way.**

Proof. Step 1: Let S_i be a component of the Apollonian gasket on (M,I). Then $S_i =$ Abs$\cap\mathbb{P}W_i$, where W_i is a rational subspace. Denote by Γ the group of all $\nu \in O^+(H^2(M,\mathbb{Z}))$ preserving the Hodge decomposition. Then $\mathcal{H} := \mathbb{P}\,\mathrm{Pos}(NS)/\Gamma$ is a hyperbolic lattice, and Aut(M,I) is the group of all $\nu \in \Gamma$ preserving the Hodge decomposition. Let $\Gamma_{W_i} := \{\gamma \in \Gamma \mid \Gamma(W_i) = W_i\}$. Then Γ_{W_i} is a lattice in $O(W_i)$, hence it acts on S_i with dense orbits. This implies that S_i wholly belongs to the closure Kähler cone, and it cannot transversally intersect the orthogonal complement to a (-2)-class. Indeed, the the orthogonal complements to (-2)-classes are faces of the Kähler cone, and S_i does not cross these faces.

Step 2: Let $U_1, U_2 \subset V$ be spaces of negative signature of a space of signature $(1, n)$. Then $\mathbb{P}\operatorname{Pos} U_1^\perp$ intersects $\mathbb{P}\operatorname{Pos} U_2^\perp$ if and only if $U_1 + U_2$ is negative definite. If $U_1 + U_2$ is degenerate, the spheres $\operatorname{Abs}\mathbb{P}\operatorname{Pos} U_1^\perp$ and $\operatorname{Abs}\mathbb{P}\operatorname{Pos} U_2^\perp$ are tangent; otherwise, these spheres do not intersect either. **Therefore, S_i does not transversally intersect an orthogonal complement to a (-2)-class if and only if $W_i^\perp + \eta$ is not negative definite for any (-2)-class η not orthogonal to W_i.**

Step 3: In this case S_i belongs to a Weyl chamber, and Γ_{W_i} preserves this Weil chamber, because it preserves S_i. **Therefore, $\Gamma_{W_i} \subset \operatorname{Aut}(M)$.** Since S_i is the limit set of Γ_{W_i}, this implies that S_i belongs to the Apollonian gasket.

Step 4: Conversely, assume that $W \subset NS_{\mathbb{R}}$ be a rational subspace of signature $(1, k)$ such that $\operatorname{Pos} W$ does not intersect $\rho^\perp$ for all (-2)-classes. Consider a subgroup $\Gamma_W^\circ \subset \Gamma_W \subset \Gamma$ consisting of all elements of Γ_W acting trivially on $W^\perp$. Then Γ_W° has finite index in $O(W_{\mathbb{Z}})$. Since Γ_W° preserves its limit set, which belongs to $\operatorname{Abs}\operatorname{Kah}_{\mathbb{Q}}$, it actually preserves the ample cone, hence $\Gamma_W^\circ \subset \operatorname{Aut}(M)$. **Therefore, its limit set belongs to the limit set of $\operatorname{Aut}(M)$.**

■

Baragar gasket

REMARK: Let $L \subset \overline{\mathsf{Kah}}_{\mathbb{Q}}$ be a face of a Kähler cone which does not meet faces of smaller dimension, $\dim L > 1$. **Then $\mathbb{P}\,\mathsf{Pos}\,\mathbb{L} \cap \mathsf{Abs}$ is a component of the Apollonian gasket**.

DEFINITION: The union of all components of the Apollonian gasket obtained this way is called **the Baragar gasket**.

REMARK: The Baragar components do not intersect transversally, but they might be tangential. **Other components of the Apollonian gasket can intersect,** but they cannot intersect the Baragar components transversally.