Automorphisms of hyperkahler manifolds and fractal geometry of limit sets of hyperbolic groups

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Hyperbolic manifolds

REMARK: $SO^+(p,q)$ denotes the connected component of the orthogonal group $SO(p,q)$

DEFINITION: Hyperbolic space is the quotient $SO^+(1,n)/SO(n)$ equipped with an $SO^+(1,n)$-invariant Riemannian metric (which is unique up to a scalar multiplier).

DEFINITION: A Kleinian group is a discrete subgroup $\Gamma \subset SO^+(1,n)$ of finite Haar covolume (that is, **the quotient $SO^+(1,n)/\Gamma$ has finite volume**).

DEFINITION: Let V be a real space equipped with a quadratic form of signature $(1,n)$. Let $\mathbb{P}^+(V)$ be the projectivisation of a positive cone. We identify $\mathbb{P}^+(V)$ with the hyperbolic space $SO^+(1,n)/SO(n)$. A **hyperbolic orbifold** is a quotient of $\mathbb{P}^+(V)$ by a Kleinian subgroup of $SO(V)$.

THEOREM: Any finite volume complete Riemannian orbifold of constant negative sectional curvature is obtained this way.

Limit sets

DEFINITION: An **arithmetic subgroup** of an algebraic group G is a finite index subgroup in $G_{\mathbb{Z}}$.

REMARK: From Borel and Harish-Chandra, it follows that **any arithmetic subgroup of $SO(1,n)$ is Kleinian, for $n \geqslant 2$.**

DEFINITION: The **absolute** of $\mathbb{H}^n = \mathbb{P}^+(V)$ is the set $\operatorname{Abs} := \mathbb{P}(\{\eta \in V \ \mid \ q(\eta,\eta) = 0\})$, identified with the Euclidean sphere.

DEFINITION: Given a group $\Gamma \subset SO(1,n)$ acting on $\mathbb{H}^n$, its **limit set** is the set of all $x \in \operatorname{Abs}$ obtained as limit points of an orbit of Γ.

REMARK: It does not matter which orbit we take, **the limit set of $\Gamma \cdot x$ does not depend on the choice of $x \in \mathbb{H}^n$.**

REMARK: **The limit set of a Kleinian group is always the whole** Abs (it follows, for instance, from Ratner's theorem on classification of orbits).

CHOBPs and their boundary

DEFINITION: Let $\mathcal{H} = \mathbb{H}^n/\Gamma$ be a hyperbolic orbifold, and $S_1, ..., S_n \subset \mathcal{H}$ a collection of closed locally geodesic immersed hypersurfaces. A **convex hyperbolic orbifold with polyhedral boundary** (CHOPB) is a connected component of the complement $\mathcal{H}\backslash \bigcup_i S_i$.

DEFINITION: Let $\mathcal{P}$ be a CHOPB, and $P \subset \mathbb{H}^n$ a connected component of its preimage in $\mathbb{H}^n$. **The isotropic boundary** of the CHOPB is the set $\overline{P} \cap \mathrm{Abs}$, where $\mathrm{Abs} = \partial\mathbb{H}^n$.

REMARK: It turns out that the isotropic boundary of a CHOPB **is either a countable set or a fractal;** in particular, **its interior is empty, unless $P = \mathbb{H}^n$.**

THEOREM: Let $V_{\mathbb{Z}}$ be a lattice equipped with an integer-valued quadratic form of signature $(1, n)$, $\Gamma \subset SO^+(V_{\mathbb{Z}})$ a finite index subgroup, and $\mathcal{P} \subset \mathcal{H} = \mathbb{H}^n/\Gamma$ a CHOPB. **Then all irrational points on the isotropic boundary $\overline{P} \cap \mathrm{Abs}$ belong to the limit set of $\pi_1(\mathcal{P})$, acting on $\overline{P}$ in a natural way.**

PROPOSITION: Let $\Gamma_0 \subset SO(1, n)$ be a subgroup, and $\Lambda \subset \mathrm{Abs}$ its limit set. Then **all orbits of Γ_0-action on Λ are dense in Λ.** In other words, **Λ is a fractal.**

CHOBPs and Apollonian carpets

DEFINITION: Let $\Gamma \subset SO(1,n)$ be a discrete subgroup, and $S_1, ..., S_k \subset \operatorname{Abs}$ a finite collection of spheres of any dimension. An **Apollonian carpet** is a closure of $\cup_i \Gamma \cdot S_i$.

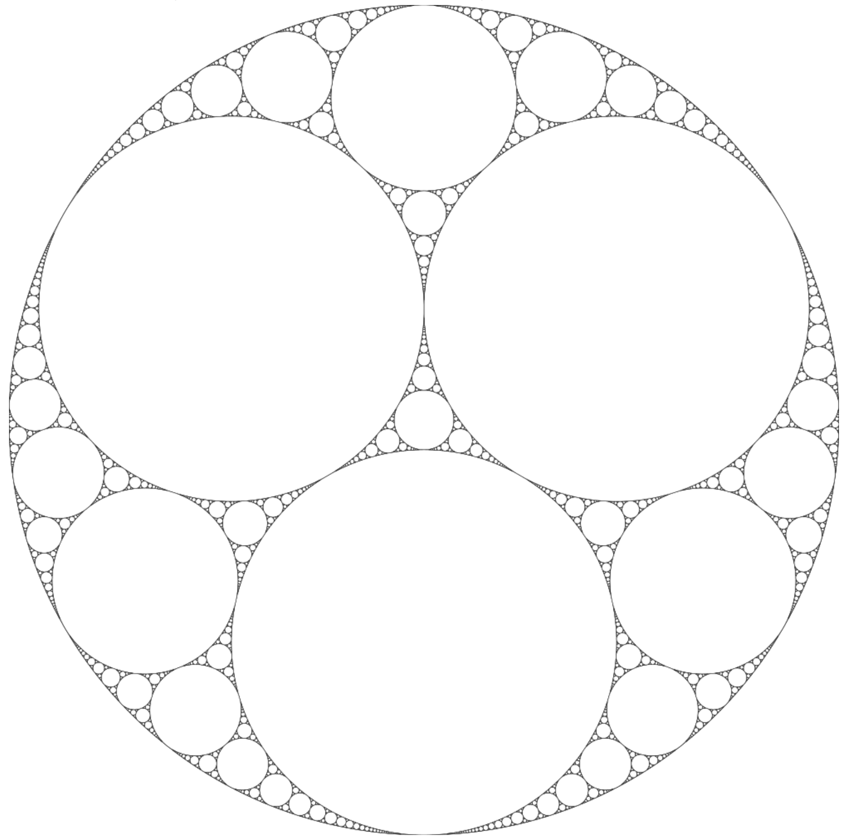
THEOREM: Let $\mathcal{P} \subset \mathcal{H} = \mathbb{H}^n/\Gamma$ be a CHOPB, $P \subset \mathbb{H}^n$ a connected component of its preimage, $\Gamma_0 = \pi_1(\mathcal{P})$ its fundamental group, and $B := \operatorname{Abs} \cap \overline{P}$ its isotropic boundary. Then **either B does not contain a real analytic interval or B is a union of an Apollonian carpet and a countable set.**

REMARK: This result **follows from N. Shah's theorem about the closure of a geodesic ray in a hyperbolic manifold.**

REMARK: Let $B_R \subset \mathbb{H}^n$ be a geodesic ball of radius R. **The growth rate of the function $F(R) := \#(\Gamma_0 x \cap B_R)$ is expressed in terms of the Hausdorff dimension of the limit set Λ** (Sullivan). This gives an asymptotics of the number of of rational curves of a given degree on a K3 surface (Dolgachev).

Apollonian gasket

Apollonian gasket is obtained by taking three kissing circles, inscribing a fourth circle kissing these three, then inscribing a circle inside a triangle formed by the segments of three circles, and so on, ad infinitum:



This construction can be performed on a 2-sphere. Since the Moebius group acts transitively on triples of pairwise kissing circles, **the Apollonian gasket on $\mathbb{C}P^1$ is unique up to a conformal equivalence.**

Apollonian gasket (2)

The Apollonian gasket **is a limit set of a discrete subgroup of the Moebius group $PSL(2,\mathbb{C})$ acting on $\mathbb{C}P^1$** ("Indra's Pearls", by D. Mumford, C. Series and D. Wright).

The largest artwork in history, "Earth Drawings" by Jim Denevan, drawn in the sand of Black Rock Desert (Nevada), depicts the Apollonian gasket.

This fractal **can be obtained as the isotropic boundary of a universal covering of a CHOPB** (later today).

THE KISS PRECISE

The Apollonian gasket is a classical concept, **known since Apollonius of Perga, c. 240 BC - c. 190 BC.**

Its construction is described in a poem by Frederic Soddy (1877–1956), an English radiochemist who received the Nobel Prize for the discovery of isotopes. Soddy independently solved the Problem of Apollonius (originally proven by Descartes) about the relation between the radii of kissing circles, and published its proof in "Nature" as a poem:

THE KISS PRECISE

For pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

F. Soddy, *"The Kiss Precise,"* Nature, v. 137, p. 1021 (1936)

Apollonian carpets of K3-type

Let $(V_{\mathbb Z}, q)$ be a lattice equipped with an integer quadratic form of signature $(1, n)$. A **(-2)-vector** is a vector $v \in V_{\mathbb Z}$ such that $q(v,v) = -2$. It is known that the group $SO^+(V_{\mathbb Z})$ acts on the set of (-2)-vectors with finitely many orbits.

Let $V := V_{\mathbb Z} \otimes_{\mathbb Z} \mathbb R$ and $\mathcal H := \mathbb P V^+/SO^+(V_{\mathbb Z})$; this quotient is a hyperbolic orbifold. Denote by S the set of orthogonal complements to all (-2)-classes; the image of S in $\mathcal H$ is a finite collection of hyperbolic hypersurfaces.

DEFINITION: K3-type CHOPB is a connected component of $\mathcal H \backslash S$.

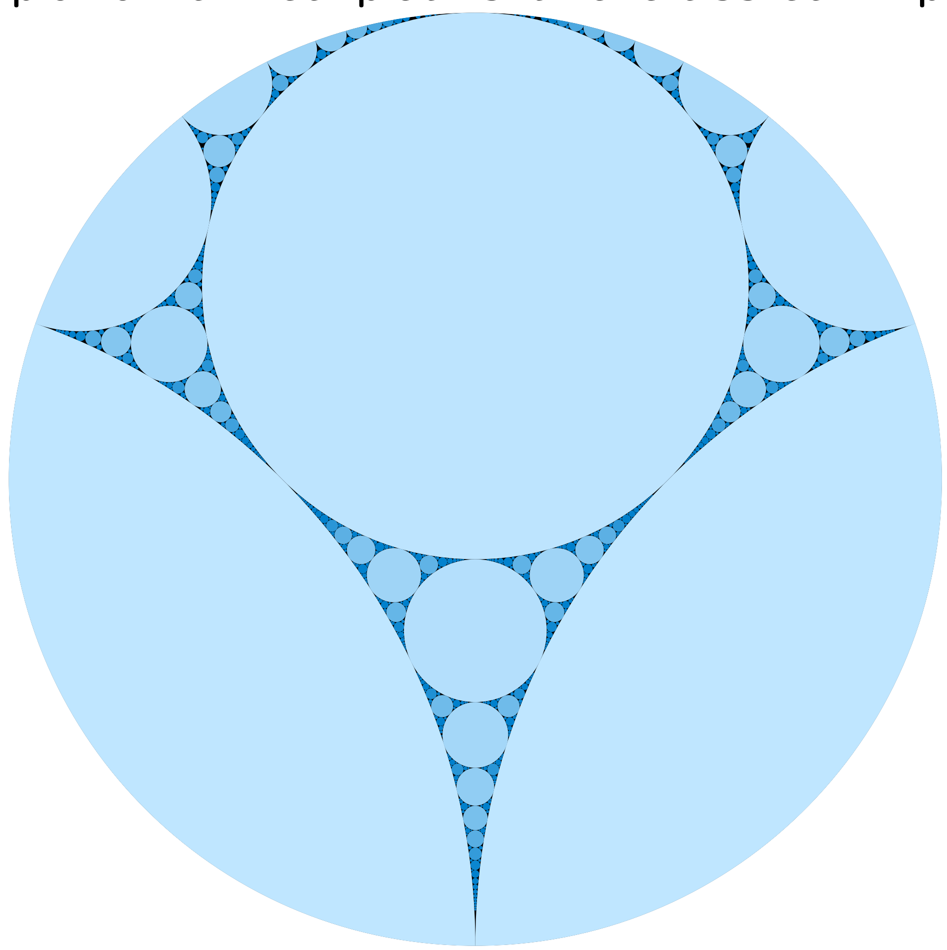
REMARK: We call this CHOPB "a K3-type CHOPB" because for any $n \leqslant 9$, **any even lattice $V_{\mathbb Z}$ can be realized as a lattice of integer (1,1)-classes in a projective K3 surface M, and** $\operatorname{Aut}(M) = \pi_1(\mathcal P)$**,** where $\mathcal P$ is a K3-type CHOPB defined above.

Apollonian gasket as a boundary of CHOPB

Consider the intersection lattice

$$\begin{pmatrix} -2 & 2 & 2 & 2 \\ 2 & -2 & 2 & 2 \\ 2 & 2 & -2 & 2 \\ 2 & 2 & 2 & -2 \end{pmatrix}$$

The corresponding Apollonian carpet is the classical Apollonian gasket:

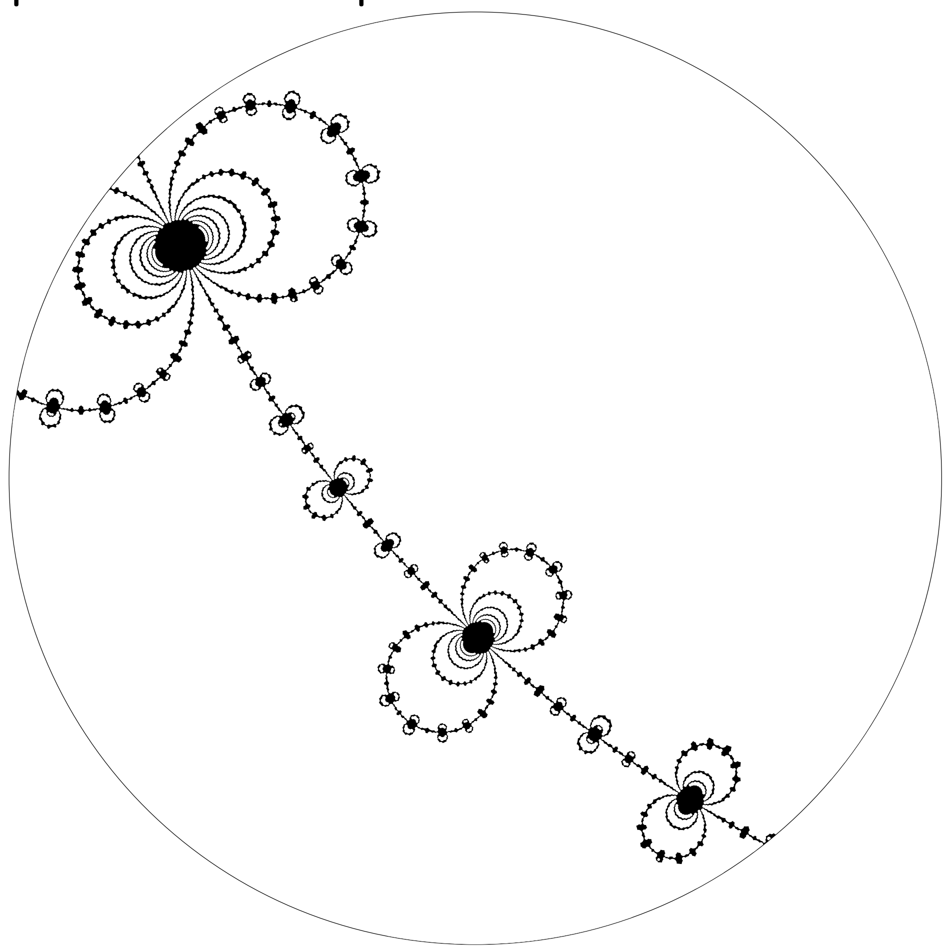


Apollonian carpet: Example 1

Consider the intersection lattice

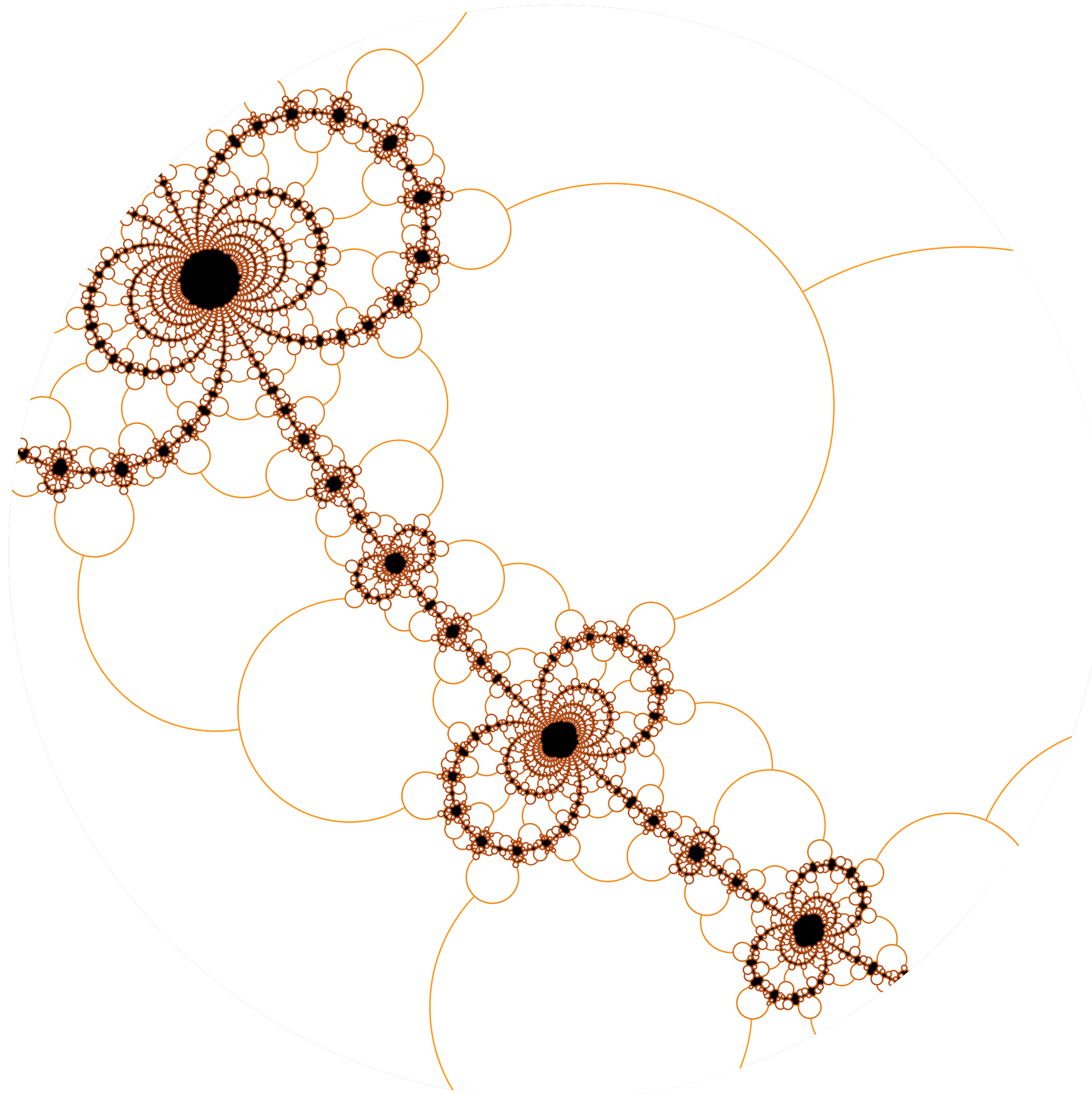
$$A = \begin{pmatrix} -2 & 4 & 0 & 0 \\ 4 & -2 & 2 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 2 & -4 \end{pmatrix}$$

The corresponding Apollonian carpet:



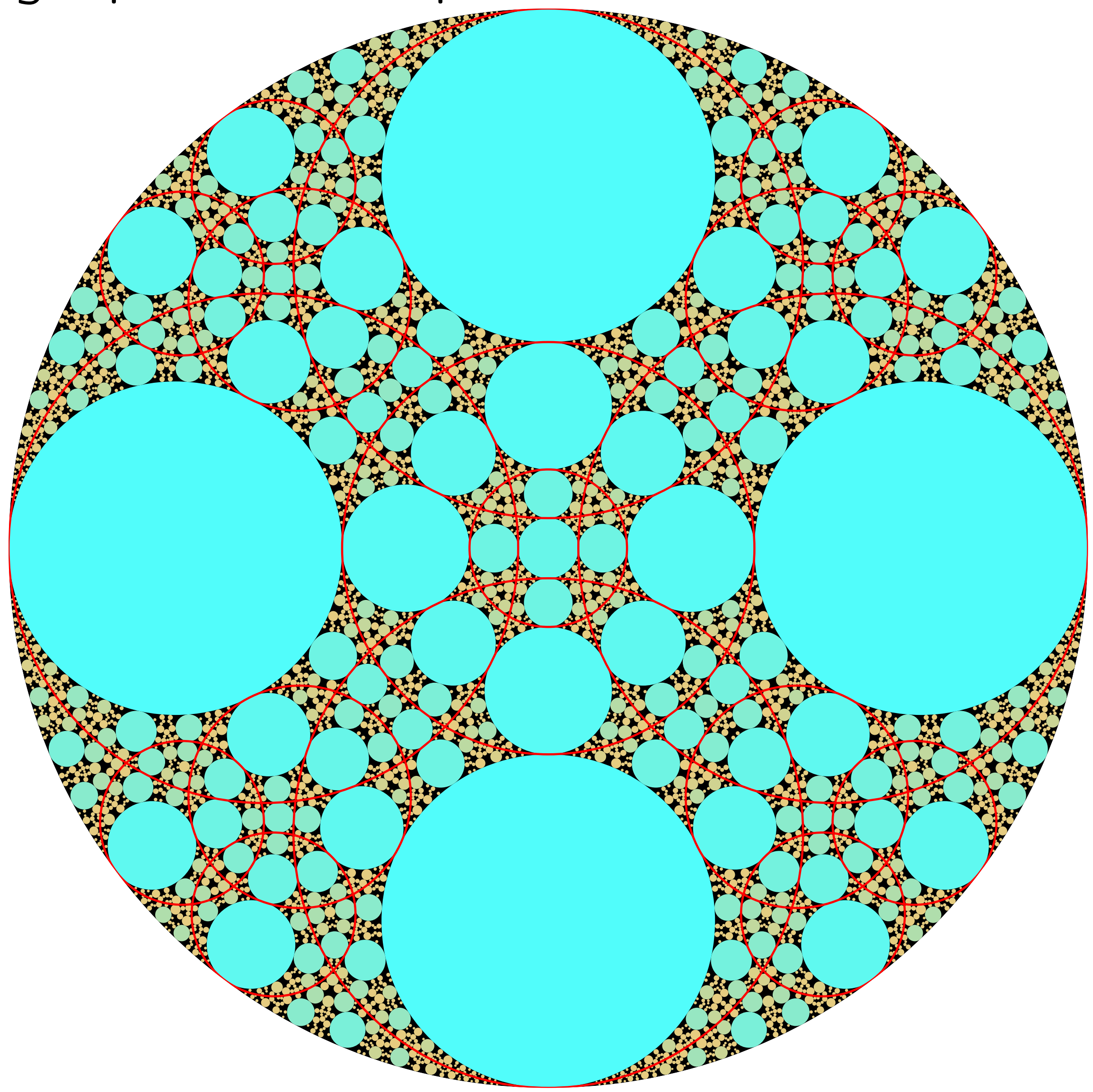
Apollonian carpet: Example 2

Consider the intersection form $q = 10x^2 - 2y^2 - 10z^2 - 10w^2$. The corresponding Apollonian carpet:



Apollonian carpet: Example 3

Consider the intersection form $q = 2p^3x^2 - 2p^2y^2 - 2pz^2 - 2w^2$, where $p = 5$. The corresponding Apollonian carpet:

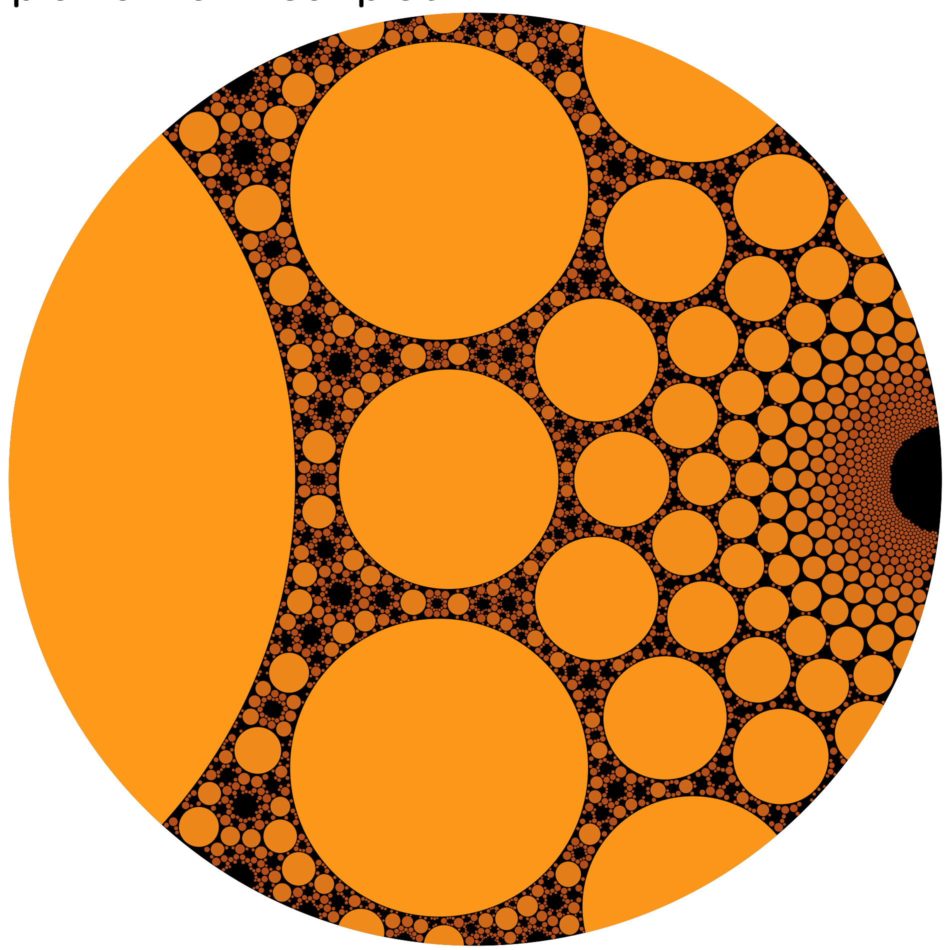


Apollonian carpet: Example 4

Consider the intersection form

$$\begin{pmatrix} -2 & 5 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 5 & -10 \end{pmatrix}.$$

The corresponding Apollonian carpet:

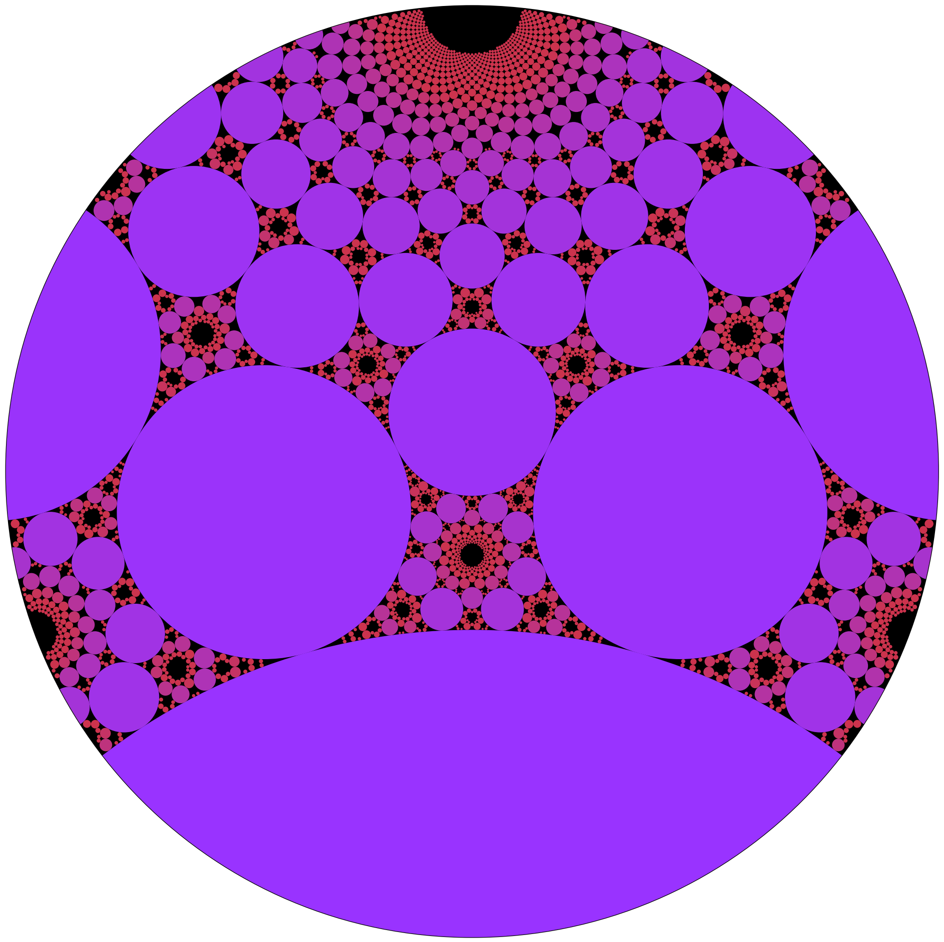


Apollonian carpet: Example 5

Consider the intersection form

$$\begin{pmatrix} -2 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

The corresponding Apollonian carpet:



Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I : TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x,y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x,y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

Kähler manifolds

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M. The set of all Kähler classes is called **the Kähler cone**.

REMARK: **(the Hodge decomposition)**
The second cohomology of a compact Kähler manifold are decomposed as $H^2(M, \mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$, where $H^{2,0}(M)$ is the space of all cohomology classes which can be represented by holomorphic $(2, 0)$-forms, $H^{0,2}(M)$ its complex conjugate, and $H^{1,1}(M)$ the classes which can be represented by I-invariant forms.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

Hyperkähler manifolds of maximal holonomy

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **maximal holonomy**, or **IHS** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold **admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.**

Further on, **all hyperkähler manifolds are assumed to be of maximal holonomy**.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.**

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{2n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(3, b_2 - 3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

Monodromy group

DEFINITION: **Monodromy group** Mon(M) of a hyperkähler manifold (M,I) is a subgroup of $O(H^2(M,\mathbb{Z}),q)$ generated by monodromy of Gauss-Manin connections for all families of deformations of (M,I). The **Hodge monodromy group** Mon(M,I) is a subgroup of Mon(M) preserving the Hodge decomposition.

THEOREM: Mon(M) **is an arithmetic subgroup of** $SO(H^2(M,\mathbb{Z}),q)$**.**

DEFINITION: Let (M,I') be a holomorphic symplectic manifold pseudo-isomorphic to (M,I). A **Kähler chamber** of (M,I) is an image of the Kähler cone of (M,I') under the action of Mon(M,I).

CLAIM: Mon(M,I) acts on $H^{1,1}(M,I)$ **mapping Kähler chambers to Kähler chambers.**

THEOREM: **The group of automorphisms** Aut(M,I) **is a group of all elements of** Mon(M,I) **preserving the Kähler cone.**

Positive cone

DEFINITION: Let P be the set of all real vectors in $H^{1,1}(M,I)$ satisfying $q(v,v)>0$, where q is the Bogomolov-Beauville-Fujiki form on $H^2(M)$. The **positive cone** $\operatorname{Pos}(M,I)$ as a connected component of P containing a Kähler form. Then $\mathbb{P}\operatorname{Pos}(M,I)$ is a hyperbolic space, and $\operatorname{Mon}(M,I)$ acts on $\mathbb{P}\operatorname{Pos}(M,I)$ by hyperbolic isometries.

THEOREM: The positive cone is partitoned onto Kähler chambers. Interiors of different Kähler chambers are disjoint, the closure of their union contains the positive cone.

DEFINITION: Let $H^{1,1}(M,\mathbb{Q})$ be the set of all rational (1,1)-classes on (M,I), and $\operatorname{Kah}_{\mathbb{Q}}(M,I)$ the set of all Kähler classes in $H^{1,1}(M,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{R}$. Then $\operatorname{Kah}_{\mathbb{Q}}(M,I)$ is called **ample cone** of M.

Hyperbolic manifolds associated with a hyperkähler manifold

REMARK: From global Torelli theorem it follows that $\operatorname{Mon}(M,I)$ is a finite index subgroup in $O(H^2(M,\mathbb{Z}),q)$. Therefore, $\operatorname{Mon}(M,I)$ **acts on** $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M,I) := \mathbb{P}(\operatorname{Pos}(M,I)\cap H^{1,1}(M,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{R})$ **with finite covolume;** in other words, $\operatorname{Mon}(M,I)$ is Kleinian, and the quotient $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M,I)/\operatorname{Mon}(M,I)$ is a finite volume hyperbolic orbifold.

REMARK: Notice that $\operatorname{Aut}(M,I)$ is a stabilizer of $\operatorname{Kah}(M)$ in $\operatorname{Mon}(M,I)$.

THEOREM: (cone conjecture; proven by Amerik-V.)
The quotient $\operatorname{Kah}_{\mathbb{Q}}(M,I)/\operatorname{Aut}(M,I)$ is a finite hyperbolic polyhedron in $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M,I)/\operatorname{Mon}(M,I)$.

REMARK: In other words, **the action of $\operatorname{Aut}(M,I)$ on $\operatorname{Kah}_{\mathbb{Q}}(M,I)$ has a finite polyhedral fundamental domain.**

MBM classes

DEFINITION: Negative class on a hyperkähler manifold is $\eta \in H_2(M,\mathbb{R}) = H^2(M,\mathbb{R})$ satisfying $q(\eta,\eta) < 0$. It is **effective** if it is represented by a curve.

DEFINITION: A negative class $z \in H^2(M,\mathbb{Z})$ on a hyperkähler manifold is called **an MBM class** if there exist a deformation of M with $\operatorname{Pic}(M) = \langle z\rangle$ such that λz is represented by a curve, for some $\lambda \neq 0$.

THEOREM: Let (M,I) be a hyperkähler manifold, and $S \subset H_{1,1}(M,I)$ the set of all MBM classes in $H_{1,1}(M,I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \ \mid \ z \in S\}$ in $H^{1,1}(M,I)$. **Then the Kähler cone of (M,I) is a connected component of $\operatorname{Pos}(M,I)\backslash \cup S^\perp$**, where $\operatorname{Pos}(M,I)$ is a positive cone of (M,I). Moreover, for any connected component K of $\operatorname{Pos}(M,I)\backslash \cup S^\perp$, there exists $\gamma \in O(H^2(M))$ in a monodromy group of M, and a hyperkähler manifold (M,I') birationally equivalent to (M,I), such that $\gamma(K)$ is a Kähler cone of (M,I').

CHOPBs and automorphisms

REMARK: This implies that **faces of the Kähler cone are orthgonal complements to MBM classes.**

REMARK: The set of MBM classes **is** $\operatorname{Mon}(M, I)$**-invariant.**

THEOREM: Consider the hyperbolic manifold $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M, I)/\operatorname{Mon}(M, I)$, and let $S \subset \mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M, I)$ be the set of all orthogonal complements to MBM classes, realized as a hyperbolic hyperspace in $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M, I)$. **Then the projectivized ample cone of (M, I) is one of the connected components of $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M, I)\backslash S$, and $\operatorname{Aut}(M) = \pi_1(\mathcal{P})$, where $\mathcal{P}$ is the corresponding CHOPB.**