Construction of automorphisms of hyperkähler manifolds

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Hyper-Kähler Manifolds and Related Structures in Algebraic and Differential Geometry

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The Kähler cone and its faces

This is **joint work with Ekaterina Amerik**.

THEOREM: Let M be a hyperkähler manifold, $\mathsf{Mon}(M)$ the group of automorphisms of $H^2(M)$ generated by monodromy transform for all Gauss-Manin local systems, and $\mathsf{Mon}_I(M)$ the Hodge monodromy group, that is, a subgroup of $\mathsf{Mon}(M)$ preserving the Hodge decomposition. **Then $\mathsf{Aut}(M)$ is a subgroup of $\mathsf{Mon}_I(M)$ preserving the Kähler cone $\mathsf{Kah}(M)$.**

Proof: Follows from Torelli theorem (this observation is due to E. Markman).

Aim of today's talk: describe $\mathsf{Kah}(M)$ and $\mathsf{Aut}(M)$ in terms of invariants of M called **MBM classes**. Give a classification of holomorphic automorphisms of M (hyperbolic, parabolic, elliptic). Prove the following theorem.

THEOREM: Let M be a hyperkähler manifold, with $b_2(M) \geqslant 5$. Then M has a deformation admitting a hyperbolic automorphism. If $b_2(M) \geqslant 7$, M has a deformation admitting a parabolic automorphism.

REMARK: By construction, these M satisfy $\mathsf{Aut}(M) = \mathsf{Mon}_I(M)$

REMARK: By ergodicity theorem, such M are also dense on the deformation space.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(3, b_2-3)$.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

Monodromy group of a hyperkähler manifold

DEFINITION: Let M be a hyperkähler manifold, and $\mathsf{Mon}(M)$ the group of automorphisms of $H^2(M)$ generated by monodromy transform for all Gauss-Manin local systems. Then $\mathsf{Mon}(M)$ is called **the monodromy group of M**.

THEOREM: (V., follows from Torelli theorem)
Then group $\mathsf{Mon}(M) \subset O(H^2(M, \mathbb{Z})$ has finite index in $O(H^2(M, \mathbb{Z}))$.

DEFINITION: **An arithmetic lattice**, or **arithmetic lattice subgroup** in an algebraic group G defined over $\mathbb{Q}$ is a finite index subgroup in $G_{\mathbb{Z}}$.

REMARK: For example, $\mathsf{Mon}(M)$ **is an arithmetic lattice in $O(H^2(M, \mathbb{Q}), q)$.**

The group of symplectic Hodge monodromy

DEFINITION: Let (M,I) be a hyperkähler manifold. Then **the Hodge monodromy group** $\mathsf{Mon}_I(M)$ is the group of all $a\in \mathsf{Mon}(M)$ preserving the Hodge decomposition on $H^2(M)$.

DEFINITION: Let Ω be a holomorphic symplectic form on a hyperkähler manifold. Consider the homomorphism $\varphi : \mathsf{Mon}_I(M) \longrightarrow \mathbb{C}^*$, $\varphi(\gamma)=\frac{\gamma^*\Omega}{\Omega}$. Denote its kernel by $\mathsf{Mon}_{I,\Omega}(M,I)$. Thi group is called **the group of symplectic Hodge monodromy.**

Claim 1: Consider the Hodge lattice $\Lambda := H^{1,1}_I(M,\mathbb{Z})$. **Then the natural homomorphism $\mathsf{Mon}_{I,\Omega}(M,I)\longrightarrow O(\Lambda)$ is injective and has finite index.**

Proof: Let $H^2_{tr}(M) := H^{1,1}_I(M,\mathbb{Q})^\perp$ be the "transcendental part" of the Hodge lattice, that is, the smallest Hodge substructure containing $\operatorname{Re} H^{2,0}(M)$. By definition,

$$\mathsf{Mon}_{I,\Omega}(M,I) = \left\{ a\in \mathsf{Mon}(M) \ \middle|\ a\Big|_{H^2_{tr}(M)} = \mathsf{Id} \right\}$$

Since $\mathsf{Mon}(M)$ is an arithmetic lattice subgroup in $O(H^2(M,\mathbb{Z}))$, $\mathsf{Mon}_{I,\Omega}(M,I)$ is arthmetic lattice in the group of isometries of $H^2_{tr}(M)^\perp = \Lambda$. ■

MBM classes

DEFINITION: **Negative class** on a hyperkähler manifold is $\eta \in H_2(M,\mathbb{R}) = H^2(M,\mathbb{R})$ satisfying $q(\eta,\eta) < 0$. It is **effective** if it is represented by a curve.

THEOREM: Let $z \in H_2(M,\mathbb{Z})$ be negative, and I, I' complex structures in the same deformation class, such that z is of type (1,1) with respect to I and I' and $\operatorname{Pic}(M) = \langle z \rangle$. Then $\pm z$ **is effective in** (M,I) $\Leftrightarrow$ **iff it is effective in** (M,I').

REMARK: From now on, we identify $H^2(M)$ and $H_2(M)$ using the BBF form. Under this identification, **integer classes in** $H_2(M)$ **correspond to rational classes in** $H^2(M)$ (the form q is not unimodular).

DEFINITION: A negative class $z \in H^2(M,\mathbb{Z})$ on a hyperkähler manifold is called **an MBM class** if there exist a deformation of M with $\operatorname{Pic}(M) = \langle z \rangle$ such that λz is represented by a curve, for some $\lambda \neq 0$.

MBM classes and the shape of the Kähler cone

THEOREM: Let (M, I) be a hyperkähler manifold, and $S \subset H_{1,1}(M, I)$ the set of all MBM classes in $H_{1,1}(M, I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \ | \ z \in S\}$ in $H^{1,1}(M, I)$. **Then the Kähler cone of (M, I) is a connected component of $\operatorname{Pos}(M, I) \backslash \cup S^\perp$**, where $\operatorname{Pos}(M, I)$ is a positive cone of (M, I). Moreover, for any connected component K of $\operatorname{Pos}(M, I) \backslash \cup S^\perp$, there exists $\gamma \in O(H^2(M))$ in a monodromy group of M, and a hyperkähler manifold (M, I') birationally equivalent to (M, I), such that $\gamma(K)$ is a Kähler cone of (M, I').

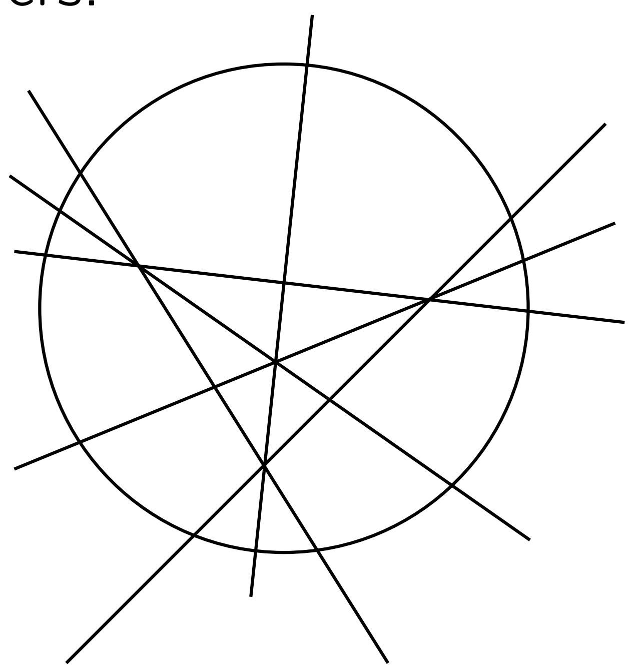
REMARK: This implies that **MBM classes correspond to faces of the Kähler cone.**

DEFINITION: **Kähler chamber** is a connected component of $\operatorname{Pos}(M, I) \backslash \cup S^\perp$.

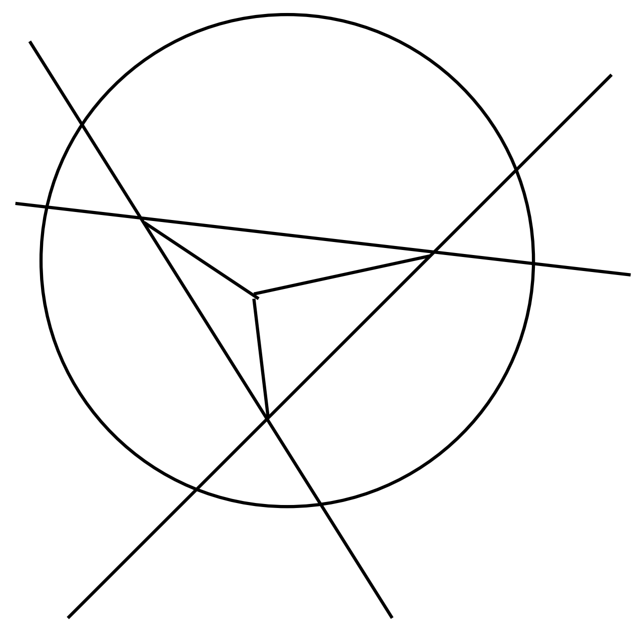
CLAIM: The Hodge monodromy group maps Kähler chambers to Kähler chambers.

MBM classes and the Kähler cone: the picture

REMARK: For any negative vector $z \in H^2(M)$, $z^\perp \cap \mathsf{Pos}(M,I)$ **either has dense intersection with the interior of the Kähler chambers (if z is not MBM), or is a union of walls of those (if z is MBM);** that is, there are no "barycentric partitions" in the decomposition of the positive cone into the Kähler chambers.



Allowed partition



Prohibited partition

MBM classes and automorphisms

THEOREM: Let (M,I) be a hyperkähler manifold, $\mathsf{Mon}(M)$ the group of automorphisms of $H^2(M)$ generated by monodromy transform for all Gauss-Manin local systems, and $\mathsf{Mon}_I(M)$ the Hodge monodromy group, that is, a subgroup of $\mathsf{Mon}(M)$ preserving the Hodge decomposition. **Then** $\mathsf{Aut}(M)$ **is a subgroup of** $\mathsf{Mon}_I(M)$ **preserving the Kähler cone** $\mathsf{Kah}(M)$**.**

Proof: Follows from Torelli theorem (this observation is due to E. Markman). ■

COROLLARY: Let (M,I) be a hyperkähler manifold such that there are no MBM classes of type (1,1). **Then** $\mathsf{Aut}(M) = \mathsf{Mon}_I(M)$**.**

Proof: Indeed, for such manifold $\mathsf{Kah}(M,I) = \mathsf{Pos}(M,I)$. ■

Morrison-Kawamata cone conjecture

DEFINITION: An integer cohomology class a is **primitive** if it is not divisible by integer numbers $c > 1$.

THEOREM: (a version of Morrison-Kawamata cone conjecture) **The group $\operatorname{Mon}(M)$ acts on the set of primitive MBM classes with finitely many orbits.**

Proof: Proven by Amerik-V., using homogeneous dynamics (Ratner theorems, Dani-Margulis, Mozes-Shah). ■

COROLLARY: Let M be a hyperkähler manifold. Then there exists a number $N > 0$, called **MBM bound**, such that any MBM class z satisfies $|q(z, z)| < N$.

Proof: There are only finitely many primitive MBM classes, up to isometry action, and the have finitely many squares. ■

Corollary 1: Let M be a hyperkähler manifold, N its MBM bound, and (M, I) a deformation such that for any $x \in H^{1,1}_I(M, \mathbb{Z})$ one has $q(x, x) > N$. **Then (M, I) has no MBM classes of type (1,1), and $\operatorname{Kah}(M, I) = \operatorname{Pos}(M, I)$ and $\operatorname{Aut}(M) = \operatorname{Mon}_I(M)$.** ■

Classification of automorphisms of hyperbolic space

REMARK: The group $O(m,n), m,n > 0$ has 4 connected components. We denote the connected component of 1 by $SO^+(m,n)$. We call a vector v **positive** if its square is positive.

DEFINITION: Let V be a vector space with quadratic form q of signature $(1,n)$, $\operatorname{Pos}(V) = \{x \in V \mid q(x,x) > 0\}$ its **positive cone**, and $\mathbb{P}^+V$ projectivization of $\operatorname{Pos}(V)$. Denote by g any $SO(V)$-invariant Riemannian structure on $\mathbb{P}^+V$. Then $(\mathbb{P}^+V, g)$ is called **hyperbolic space**, and the group $SO^+(V)$ **the group of oriented hyperbolic isometries**.

Theorem-definition: Let $n > 0$, and $\alpha \in SO^+(1,n)$ is an isometry acting on V. Then one and only one of these three cases occurs

(i) α has an eigenvector x with $q(x,x) > 0$ (α is **"elliptic isometry"**)

(ii) α has an eigenvector x with $q(x,x) = 0$ and eigenvalue λ_x satisfying $|\lambda_x| > 1$ (α is **"hyperbolic isometry"**)

(iii) α has a unique eigenvector x with $q(x,x) = 0$. (α is **"parabolic isometry"**)

DEFINITION: An automorphism of a hyperkähler manifold (M,I) is called **elliptic (parabolic, hyperbolic)** if it is elliptic (parabolic, hyperbolic) on $H^{1,1}_I(M,\mathbb{R})$.

Primitive sublattices with MBM bound

DEFINITION: **Integer lattice**, or **quadratic lattice**, or just **lattice** is $\mathbb{Z}^n$ equipped with an integer-valued quadratic form. **When we speak of embedding of lattices, we always assume that they are compatible with the quadratic form.**

DEFINITION: A sublattice $\Lambda' \subset \Lambda$ is called **primitive** if $(\Lambda' \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \Lambda = \Lambda'$. A number a is **represented** by a lattice (Λ, q) if $a = q(x,x)$ for some $x \in \Lambda$. **Minumum** of a lattice is the number $\min \Lambda := \min_x |q(x,x)|$, taken over all $x \in \Lambda$ with $q(x,x) \neq 0$.

Theorem 1: Let (Λ, q) be a lattice of signature (n,m). Fix a number $N > 0$. **Then there exists a primitive sublattice $\Lambda' \subset \Lambda$ of corank 2 with $\min \Lambda' > N$.**

Proof: Later today, if time permits.

DEFINITION: Let M be a hyperkähler manifold, $\Lambda = H^2(M, \mathbb{Z})$, q the BBF form. A primitive sublattice $\Lambda' \subset H^2(M, \mathbb{Z})$ **satisfies MBM bound** if its minimum is $> N$, where N is the MBM bound of M.

Sublattices with MBM bound and automorphisms

REMARK: By Torelli theorem, **for any primitive sublattice $\Lambda \subset H^2(M,\mathbb{Z})$, there exists a complex structure I such that $\Lambda = H^{1,1}_I(M,\mathbb{Z})$.**

THEOREM: Let M be a hyperkähler manifold, and $\Lambda \subset H^2(M,\mathbb{Z})$ a primitive sublattice satisfying the MBM bound. Let (M,I) be a deformation of M such that $\Lambda = H^{1,1}_I(M,\mathbb{Z})$. **Then the group of holomorphic symplectic automorphisms $\operatorname{Aut}(M,\Omega) = \operatorname{Mon}_{I,\Omega}(M)$ is an subgroup of finite index in $O(\Lambda)$.**

Proof: Since $\Lambda = H^{1,1}_I(M,\mathbb{Z})$ satisfies MBM bound, it contains no MBM classes. **By Corollary 1, this gives $\operatorname{Aut}(M,\Omega) = \operatorname{Mon}_{I,\Omega}(M)$.** Now, $\operatorname{Mon}_{I,\Omega}(M)$ is a finite index subgroup in $O(\Lambda)$, as follows from Claim 1. ■

Existence of hyperbolic automorphisms

THEOREM: Let M be a hyperkähler manifold, with $b_2(M) \geqslant 5$. Then M has a deformation admitting a hyperbolic automorphism.

Proof. Step 1: We show that $\operatorname{Aut}(M, \Omega)$ **contains hyperbolic or parabolic elements.** Let $\Lambda = H^{1,1}_I(M, \mathbb{Z})$ be a primitive lattice of corank 2 in $H^2(M, \mathbb{Z})$ satisfying the MBM bound. Then $\operatorname{Aut}(M, \Omega)$ has finite index in $O(\Lambda)$. To simplify the argument, we replace Λ by a sublattice of smaller rank, obtaining a lattice of signature $(1, n)$.

An operator norm of an elliptic element is 1, hence any subgroup of $O(1, n)$ containing only elliptic elements has compact closure. The volume of $O(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})/O(\Lambda)$ is finite by Borel and Harish-Chandra theorem when $\operatorname{rk} \Lambda \geqslant 3$. Therefore, $O(\Lambda)$ is infinite, and $\operatorname{Aut}(M, \Omega)$ is also infinite. We obtain that $\operatorname{Aut}(M, \Omega)$ has hyperbolic or parabolic elements.

Step 2: Product of two non-commuting parabolic isometries is hyperbolic. **It remais to show that whenever there is one parabolic element p of** $\operatorname{Aut}(M, \Omega)$**, there are two such elements which do not commute.**

Step 3: Zariski closure of $O(\Lambda)$ is $O(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ by another application of Borel and Harish-Chandra. Therefore, thee the centralizer of p is an infinite index subgroup of $O(\Lambda)$. This implies that there exists $x \in \operatorname{Aut}(M, \Omega)$ such that xpx^{-1} does not commute with p. **Then $pxpx^{-1}$ is necesarily hyperbolic.** ■

Existence of parabolic automorphisms

THEOREM: Let M be a hyperkähler manifold with $b_2(M) \geqslant 7$. Then M has a deformation admitting a parabolic automorphism.

Proof. Step 1: Let $\Lambda = H^{1,1}_I(M, \mathbb{Z})$ be a primitive lattice of corank 2 in $H^2(M, \mathbb{Z})$ satisfying the MBM bound. Then $\mathrm{Aut}(M, \Omega)$ has finite index in $O(\Lambda)$. **It suffices to show that the Lie group $O(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ contains a rational unipotent subgroup U.** Then $U \cap \mathrm{Aut}(M, \Omega)$ is Zariski dense in U by another application of Borel and Harish-Chandra, and all its elements are parabolic.

Step 2: Suppose that there exists a rational vector v with $q(v, v) = 0$, and let $P \subset O(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ be the stabilizer of v. **This subgroup is clearly rational and parabolic; its unipotent radical is the group U which we require.**

Step 3: Such a rational vector exists for any indefinite lattice of rank $\geqslant 5$ by Meyer's theorem. ■

Markoff chain theorem

REMARK: A lattice (Λ, q) is **primitive** if there are no lattices $\Lambda' \subsetneq \Lambda$ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. This is the same as q being not divisible by an integer $c > 1$. Two lattices are **commensurable** if they can be embedded to the same rational lattice.

THEOREM: For each $N > 0$, there exists a primitive lattice Λ of rank 2 and signature (2,0) or (1,1) with $\min \Lambda > N$.

Proof: For signature (1,1) it's *Cassels, J. W. S., The Markoff chain. Ann. of Math. (2) 50, (1949), 676-685; A. Markoff, Math. Ann. 15, 381-406 (1879); A. Markoff, Math. Ann. 17, 379-399 (1880)*, or `http://mathoverflow.net/questions/215636/2-dimensional-sublattices-with-all-vectors-having-very-big-square-in-absolute-v/` (reply by Noam Elkies). For signature (2,0) it's obvious. ■

Proof of Theorem 1

Theorem 1: Let (Λ, q) be a lattice of signature (n, m). Fix a number $N > 0$. **Then there exists a primitive sublattice $\Lambda' \subset \Lambda$ of corank 2 with $\min \Lambda' > N$.**

Now, Theorem 1 is implied by the following theorem.

Theorem 2: Let Λ', Λ be lattices of signature (n, m) and (n', m'), with $n' \leqslant n, m' \leqslant m$. **Then there exist lattices Λ_1, Λ_1' containing Λ', Λ as corank one sublattices, such that $\Lambda_1' \otimes_{\mathbb{Z}} \mathbb{Q}$ can be embedded to $\Lambda_1 \otimes_{\mathbb{Z}} \mathbb{Q}$.**

Theorem 2 implies Theorem 1: Using Markoff Chain theorem, choose a lattice Λ' with approriate signature and $\min \Lambda' \geqslant N$, and find an embedding $\Phi: \Lambda_1' \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow \Lambda_1 \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $\Lambda_0 := \Phi(\Lambda') \cap \Lambda$ is a sublattice of corank 2 in Λ, which also satisfies $\min(\Lambda_0) > N$. ■

Rational quadratic forms: local invariants and Hasse's principle

DEFINITION: A lattice over $\mathbb{Q}$ is a vector space V over $\mathbb{Q}$ equipped with a quadratic form q.

REMARK: Clearly, V **always admits a basis $x_1, ..., x_n$ which is orthogonal with respect to q:** $q(x_i, x_j) = 0$ for $i \neq j$.

DEFINITION: Let $q = \sum a_i x_i^2$ be a quadratic form. We associate to q the following "local invariants":

- **The discriminant**: $\operatorname{disc}(q) \in \mathbb{Q}^*/(\mathbb{Q}^2)^*$, represented by $\prod a_i$.
- **p-adic invariants**: $\varepsilon_p(q) = \prod_{i<j} (a_i, a_j)_p$, where $(\cdot, \cdot)_p$ is the Hilbert symbol.
- **Signature**.

THEOREM: (Hasse's local to global principle) Two lattices over $\mathbb{Q}$ are equivalent if and only if their local invariants are equal.

Making rational quadratic form standard by adding a variable

Theorem 2: Let Λ', Λ be lattices of signature (n,m) and (n',m'), with $n' \leqslant n, m' \leqslant m$. **Then there exist lattices Λ_1, Λ_1' containing Λ', Λ as corank one sublattices, such that $\Lambda_1' \otimes_{\mathbb Z} \mathbb Q$ can be embedded to $\Lambda_1 \otimes_{\mathbb Z} \mathbb Q$.**

Theorem 2 would follow immediately if we prove the following theorem.

THEOREM: Let (V,q) be a lattice over $\mathbb Q$, with q diagonal, $q = \sum a_i x_i^2$. **Then there exists a number $t \in \mathbb Q$ such that the diagonal form $q_1 = \sum a_i x_i^2 + ty^2$ on $V_1 = V \oplus \mathbb Q$ is equivalent to the standard one $q_{st} = \sum \pm z_i^2$.**

Proof: By Hasse's principle, this would follow if

$$\varepsilon_p(q_1) = \prod_{i<j} (a_i, a_j)_p \prod (a_i, t)_p = 1 \quad (*)$$

for all p. Since Hilbert symbol is multiplicative, $\prod (a_i, t)_p = (\prod a_i, t)_p$, and (*) becomes

$$\varepsilon_p(q) = (\prod a_i, t)_p. \quad (**)$$

Since Hilbert symbol $(a,b)_p$ is 1 for all p except a finite number, (**) is a system of equations modulo a finite number of p which can be solved using Chinese reminders theorem. ∎