Constructing automorphisms of hyperkähler manifolds

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Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **of maximal holonomy**, or **IHS**, if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be of maximal holonomy.

Existence of automorphisms

Aim of today's talk: describe the automorphism group $\mathrm{Aut}(M)$ of a hyperkähler manifold in terms of invariants of M called **MBM classes**. Prove the following theorem.

THEOREM: **Let M be a hyperkähler manifold, with $b_2(M) \geqslant 5$. Then M has a deformation admitting an automorphism of infinite order,** acting on $H^{1,1}(M)$ with real eigenvalues α, β, $\alpha < 1 < \beta$ **("hyperbolically").**

This is **joint work with Ekaterina Amerik**.

THEOREM: **Let M be a hyperkähler manifold, with $b_2(M) \geqslant 14$. Then M has a deformation admitting an authomorphism of infinite order,** acting on $H^{1,1}(M)$ unipotently **("parabolic action").**

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$,** for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = 2\int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(3, b_2 - 3)$.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

Unlike $H^2(K3,\mathbb{Z})$, **the BBF form is usually not unimodular**.

Monodromy group of a hyperkähler manifold

DEFINITION: Let M be a hyperkähler manifold, and $\operatorname{Mon}(M)$ the group of automorphisms of $H^2(M)$ generated by monodromy transform for all Gauss-Manin local systems. Then $\operatorname{Mon}(M)$ is called **the monodromy group of M**.

Theorem 1:
The group $\operatorname{Mon}(M) \subset O(H^2(M,\mathbb{Z})$ has finite index in $O(H^2(M,\mathbb{Z}))$.

THEOREM: Let M be a hyperkähler manifold, $\operatorname{Mon}(M)$ the group of automorphisms of $H^2(M)$ generated by monodromy transform for all Gauss-Manin local systems, and $\operatorname{Mon}_I(M)$ the Hodge monodromy group, that is, a subgroup of $\operatorname{Mon}(M)$ preserving the Hodge decomposition. **Then $\operatorname{Aut}(M)$ surjects to the subgroup of $\operatorname{Mon}_I(M)$ preserving the Kähler cone $\operatorname{Kah}(M)$,** and the kernel of this map is finite.

Proof: Follows from global Torelli theorem (this observation is due to E. Markman).

MBM classes

DEFINITION: Negative class on a hyperkähler manifold is $\eta \in H_2(M,\mathbb{R}) = H^2(M,\mathbb{R})$ satisfying $q(\eta,\eta) < 0$. It is **effective** if it is represented by a curve.

THEOREM: Let $z \in H_2(M,\mathbb{Z})$ be negative, and I, I' complex structures in the same deformation class, such that z is of type (1,1) with respect to I and I' and $\mathrm{Pic}(M) = \langle z \rangle$. Then $\pm z$ **is effective in** (M,I) $\Leftrightarrow$ **iff it is effective in** (M,I').

REMARK: From now on, we identify $H^2(M)$ and $H_2(M)$ using the BBF form. Under this identification, **integer classes in** $H_2(M)$ **correspond to rational classes in** $H^2(M)$ (the form q is not unimodular).

DEFINITION: A negative class $z \in H^2(M,\mathbb{Z})$ on a hyperkähler manifold is called **an MBM class** if there exist a deformation of M with $\mathrm{Pic}(M) = \langle z \rangle$ such that λz is represented by a curve, for some $\lambda \neq 0$.

MBM classes and the shape of the Kähler cone

THEOREM: Let (M,I) be a hyperkähler manifold, and $S \subset H_{1,1}(M,I)$ the set of all MBM classes in $H_{1,1}(M,I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \ | \ z \in S\}$ in $H^{1,1}(M,I)$. **Then the Kähler cone of (M,I) is a connected component of** $\operatorname{Pos}(M,I)\backslash \cup S^\perp$, where $\operatorname{Pos}(M,I)$ is a positive cone of (M,I). Moreover, for any connected component K of $\operatorname{Pos}(M,I)\backslash \cup S^\perp$, there exists $\gamma \in O(H^2(M))$ in a monodromy group of M, and a hyperkähler manifold (M,I') birationally equivalent to (M,I), such that $\gamma(K)$ is a Kähler cone of (M,I').

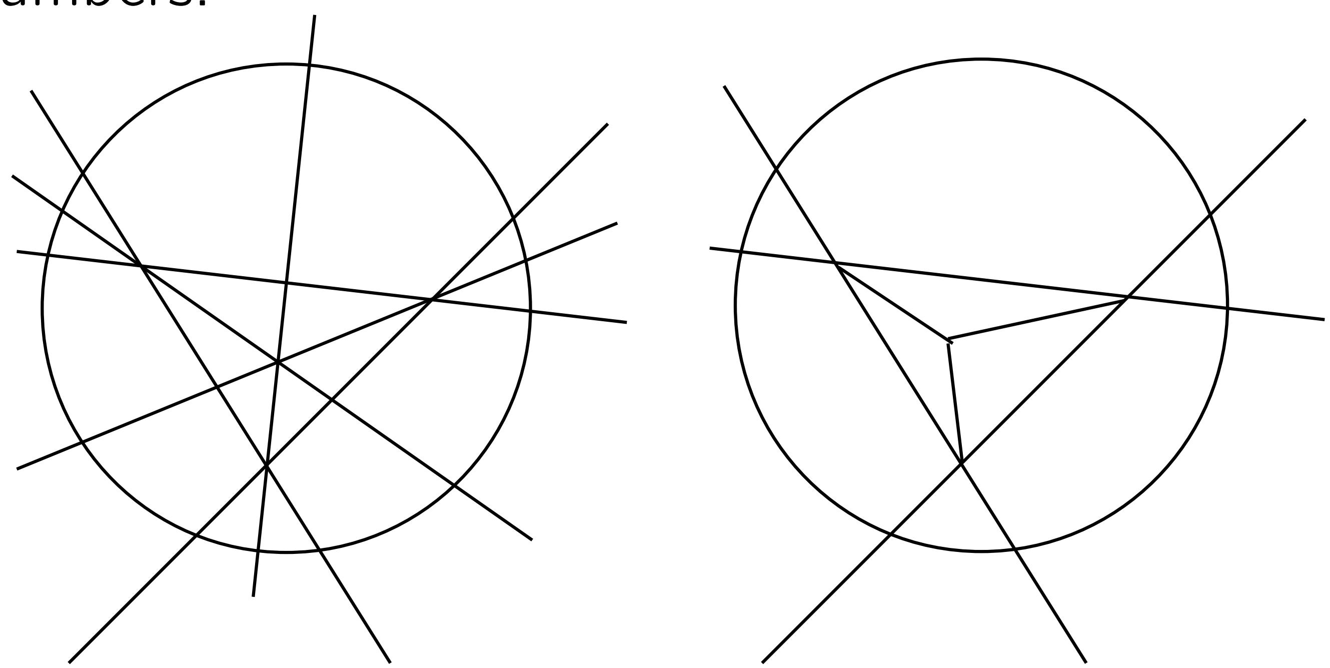
REMARK: This implies that **MBM classes correspond to faces of the Kähler cone.**

DEFINITION: **Kähler chamber** is a connected component of $\operatorname{Pos}(M,I)\backslash \cup S^\perp$.

CLAIM: The Hodge monodromy group maps Kähler chambers to Kähler chambers.

MBM classes and the Kähler cone: the picture

REMARK: For any negative vector $z \in H^2(M)$, the set $z^\perp \cap \mathrm{Pos}(M, I)$ **either has dense intersection with the interior of the Kähler chambers (if z is not MBM), or is a union of walls of those (if z is MBM);** that is, there are no "barycentric partitions" in the decomposition of the positive cone into the Kähler chambers.



Allowed partition **Prohibited partition**

MBM classes and automorphisms

THEOREM: Let (M,I) be a hyperkähler manifold, $\mathrm{Mon}(M)$ the group of automorphisms of $H^2(M)$ generated by monodromy transform for all Gauss-Manin local systems, and $\mathrm{Mon}_I(M)$ the Hodge monodromy group, that is, a subgroup of $\mathrm{Mon}(M)$ preserving the Hodge decomposition. Denote by $\mathrm{Aut}_h(M,I)$ the image of the automorphism group in $GL(H^2(M,\mathbb{R}))$. **Then $\mathrm{Aut}_h(M,I)$ is a subgroup of $\mathrm{Mon}_I(M)$ preserving the Kähler cone $\mathrm{Kah}(M,I)$.**

REMARK: The kernel of the natural map $\mathrm{Aut}(M) \longrightarrow GL(H^2(M,\mathbb{R}))$ **is a finite group which is independent from the choice of M in its deformation class.** It consists of **"absolutely trianalytic"** automorphisms of M: automorphisms which are hyperkähler in all hyperkähler structures.

COROLLARY: Let (M,I) be a hyperkähler manifold such that there are no MBM classes of type (1,1). **Then $\mathrm{Aut}(M)$ surjects to $\mathrm{Mon}_I(M)$ with finite kernel.**

Proof: Indeed, for such manifold $\mathrm{Kah}(M,I) = \mathrm{Pos}(M,I)$. ■

Morrison-Kawamata cone conjecture

DEFINITION: An integer cohomology class a is **primitive** if it is not divisible by integer numbers $c > 1$.

THEOREM: (a version of Morrison-Kawamata cone conjecture) **The group** $\operatorname{Mon}(M)$ **acts on the set of primitive MBM classes with finitely many orbits.**

Proof: Proven by Amerik-V., using homogeneous dynamics (Ratner theorems, Dani-Margulis, Mozes-Shah). ■

COROLLARY: Let M be a hyperkähler manifold. Then there exists a number $N > 0$, called **MBM bound**, such that any MBM class z satisfies $|q(z,z)| < N$.

Proof: There are only finitely many primitive MBM classes, up to isometry action, and they have finitely many squares. ■

Corollary 1: Let M be a hyperkähler manifold, N its MBM bound, and (M,I) a deformation such that for any $x \in H^{1,1}_I(M,\mathbb{Z})$ one has $q(x,x) > N$. **Then** (M,I) **has no MBM classes of type (1,1),** $\operatorname{Kah}(M,I) = \operatorname{Pos}(M,I)$**, and** $\operatorname{Aut}(M)$ **surjects to** $\operatorname{Mon}_I(M)$ **with finite kernel.** ■

Non-zero minimum of a lattice

DEFINITION: **Integer lattice**, or **quadratic lattice**, or just **lattice** is $\mathbb{Z}^n$ equipped with an integer-valued quadratic form. **When we speak of embedding of lattices, we always assume that they are compatible with the quadratic form.**

DEFINITION: A sublattice $\Lambda' \subset \Lambda$ is called **primitive** if $(\Lambda' \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \Lambda = \Lambda'$. A number a is **represented** by a lattice (Λ, q) if $a = q(x,x)$ for some $x \in \Lambda$. **Non-zero minumum** of a lattice is the number $\min_{\neq 0} \Lambda := \min_x |q(x,x)|$, taken over all $x \in \Lambda$ with $q(x,x) \neq 0$.

THEOREM: Let (Λ, q) be a lattice of signature (n,m), $n,m > 0, n+m \geqslant 5$. Fix a number $N > 0$. **Then there exists a primitive sublattice $\Lambda' \subset \Lambda$ of rank 2 with $\min_{\neq 0} \Lambda' > N$.**

Proof: Later today.

Sublattices with MBM bound and automorphisms

DEFINITION: Let M be a hyperkähler manifold, $\Lambda = H^2(M,\mathbb{Z})$, and q the BBF form. A primitive sublattice $\Lambda' \subset H^2(M,\mathbb{Z})$ **satisfies MBM bound** if its non-zero minimum is $> N$, where N is the MBM bound of M.

REMARK: By Torelli theorem, **for any primitive sublattice $\Lambda \subset H^2(M,\mathbb{Z})$, there exists a complex structure I such that $\Lambda = H^{1,1}_I(M,\mathbb{Z})$.**

THEOREM: Let M be a hyperkähler manifold, and $\Lambda \subset H^2(M,\mathbb{Z})$ a primitive sublattice satisfying the MBM bound. Let (M,I) be a deformation of M such that $\Lambda = H^{1,1}_I(M,\mathbb{Z})$. **Then the group of automorphisms $\operatorname{Aut}(M) = \operatorname{Mon}_I(M)$ surjects to a subgroup of finite index in $O(\Lambda)$.**

Proof: Since $\Lambda = H^{1,1}_I(M,\mathbb{Z})$ satisfies the MBM bound, it contains no MBM classes. **By Corollary 1, this implies that $\operatorname{Aut}(M)$ surjects to $\operatorname{Mon}_I(M)$ with finite kernel.** Now, $\operatorname{Mon}_I(M)$ surjects to a finite index subgroup in $O(\Lambda)$, as follows from Theorem 1. ■

Existence of hyperbolic automorphisms

THEOREM: Let M be a hyperkähler manifold, with $b_2(M) \geqslant 5$. Then M has a deformation admitting an automorphism of infinite order.

Proof. Step 1: Find a primitive sublattice of signature (1,1) $\Lambda \subset H^2(M,\mathbb{Z})$ satisfying the MBM bound and not representing 0. Using Torelli theorem, we construct a deformation M' of M which has $\Lambda = H^{1,1}(M') \cap H^2(M',\mathbb{Z})$.

Proof. Step 2: For such M', the symplectic automorphisms surjects to a finite index subgroup of $O(\Lambda)$.

Proof. Step 3: $O(\Lambda)$ has infinite order (follows from Dirichlet unit theorem).
■

Existence of parabolic automorphisms is proven in a similar way, but we need a primitive sublattice $\Lambda \subset H^2(M,\mathbb{Z})$ of signature (2,1) representing 0 and with $\min_{\neq 0}(\Lambda) > N$, where N is MBM bound.

Existence of sublattices

PROBLEM: Let Λ be a non-degenerate, indefinite integer lattice of signature (p,q). **Find all (p',q') such that for all such Λ there exist primitive sublattices $\Lambda' \subset \Lambda$ of signature (p',q') and with arbitrary high $\min_{\neq 0}(\Lambda)$.**

REMARK: Meyer's theorem implies that **any indefinite lattice of rank $\geqslant 5$ represents 0.** Therefore, this question is not very interesting for the usual minimum, but it becomes highly non-trivial for $\min_{\neq 0}$.

PROPOSITION: Let Λ be a non-degenerate, indefinite integer lattice $\operatorname{rk}\Lambda \geqslant 5$, and $N > 0$ any number. **Then Λ contains a primitive sublatice of signature $(1,1)$ with $\min_{\neq 0}(\Lambda) > N$.**

Proof: We use the following elementary lemma.

Lemma 1: Let (Λ, q) be a diagonal rank 2 lattice with diagonal entries α_1, α_2 divisible by an odd power of p, $\alpha_i = \beta_i p^{2n_i+1}$, such that the numbers β_i are not divisible by p and the equation $\beta_1 x^2 + \beta_2 y^2 = 0$ has no solutions modulo p. Let $v \in \Lambda \otimes \mathbb{Q}$ be any vector such that $q(v,v)$ is an integer. **Then this integer is divisible by p.**

Existence of sublattices (2)

PROPOSITION: Let Λ be a non-degenerate, indefinite integer lattice with $\operatorname{rk}\Lambda \geqslant 5$, and $N > 0$ any number. **Then Λ contains a primitive sublatice of signature $(1,1)$ with $\min_{\neq 0}(\Lambda) > N$.**

Proof. Step 1: By Meyer's Theorem, Λ has an **isotropic vector** (that is, a vector v with $q(v) = 0$). The **isotropic quadric** $\{v \in L \mid q(v) = 0\}$ has infinitely many points if it has one, and not all of them are proportional. Take two of such non-proportional points v and v', and let $v_1 := av + bv'$. Then $q(v_1) = 2abq(v, v')$. **We may chose $2ab$ to be of any sign and such that it has arbitrary large prime divisors in odd powers.**

Step 2: It is always possible to find a vector $w \in \langle v, v' \rangle^\perp$ such that $q(w)$ is divisible by an odd power of a suitable sufficiently large prime number p. Now **choose the multipliers a, b in such a way that the lattice $\Lambda' := \langle v_1, w \rangle$ satisfies assumptions of Lemma 1 and has signature (1,1).** ■

REMARK: The lattice Λ' does not represent 0, because it represents only mumbers which are divisible by odd powers of p.

Existence of sublattices (3)

THEOREM: (Nikulin, Witt...) Let Λ be a unimodular lattice of signature (p, q), and Λ' any lattice of signature (p', q') such that $2p' \leqslant p, 2q' \leqslant q$. **Then Λ' admits a primitive embedding to Λ.**

REMARK: Since one could take the quadratic form on Λ', say, $10^{100}q_0$, this theorem gives partial solution to our problem for unimodular lattices. Two caveats: (a) it does not work for non-unimodular Λ and (b) the bound $1/2\,\mathrm{rk}\,\Lambda$ is a bit too high.

To apply it to our case, we find a rational embedding of a non-unimodular Λ to $\Lambda_1 \otimes_{\mathbb{Z}} \mathbb{Q}$, where Λ_1 is a diagonal lattice with eigenvalues $\pm q$. This is possible to do using Hilbert symbols and classification of rational lattices, with $\mathrm{rk}\,\Lambda_1 = \mathrm{rk}\,\Lambda + 3$. Then one takes a primitive sublattice $\Lambda' \subset \Lambda_1$ and its intersection with Λ has arbitrarily big $\min_0$.

This makes an embedding from Λ_0 of signature (1,2) to Λ of signature $(3, 11)$ – clearly non-optimal.

QUESTION: Is it possible to optimize this construction?