

Tian-Todorov for non-Kähler holomorphically symplectic manifolds

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Bogomolov-Tian-Todorov

DEFINITION: A **holomorphic symplectic form** on a complex manifold is a holomorphic, non-degenerate 2-form Ω .

REMARK: The top power of Ω is a holomorphic volume form. Therefore, **any holomorphically symplectic manifold has trivial canonical bundle.**

DEFINITION: A manifold with trivial canonical bundle is called **Calabi-Yau**.

THEOREM: (Bogomolov-Tian-Todorov) The Kuranishi deformation space of complex structures on a compact, Kähler Calabi-Yau manifold M **is smooth**, and **its tangent space is $H^1(TM)$** , that is, the deformations of complex structures on M are **unobstructed**

Bogomolov-Tian-Todorov on holomorphically symplectic manifolds

REMARK: Non-Kähler Calabi-Yau manifolds **can have obstructed deformations**. In *É. Ghys, Déformations des structures complexes sur les espaces homogènes de $SL(2, \mathbb{C})$, *J. Reine Angew. Math.* **468** (1995), 113–138*, it was shown that **the deformation space of a locally homogeneous manifold $SL(2, \mathbb{C})/\Gamma$ can be obstructed**, for a cocompact and discrete subgroup $\Gamma \subset SL(2, \mathbb{C})$.

QUESTION: Is there any compact, simply connected holomorphically symplectic non-Kähler manifold with obstructed deformations?

The main result of today's talk: **Bogomolov-Tian-Todorov holds for a class of holomorphically symplectic non-Kähler manifolds, called Bogomolov-Guan manifolds.**

Hilbert schemes

DEFINITION: Hilbert scheme of points on a variety M is the Hilbert space of ideal sheaves $I \subset \mathcal{O}_M$ with $F := \mathcal{O}_M/I$ supported in a finite subset of M ; dimension of $H^0(F)$ is called **length**. Hilbert scheme of points for length n is denoted by $M^{[n]}$

REMARK: When M is a complex surface, $M^{[n]}$ is a smooth resolution of the n -th symmetric power of M , denoted $M^{(n)}$.

REMARK: If the surface M is holomorphically symplectic, $M^{[n]}$ is also holomorphically symplectic (follows easily from Serre's duality).

REMARK: Hilbert scheme of a K3 surface is a simply connected, holomorphically symplectic manifold. Hilbert scheme of a torus T is not simply connected, but the fiber of its Albanese map $T^{[n]} \rightarrow T$ has finite fundamental group. The universal cover of this fiber is called **generalized Kummer variety**.

This way one obtains **two main examples of simply connected holomorphically symplectic manifolds**.

Kodaira-Thurston surface

REMARK: A. Todorov conjectured that any compact, simply connected holomorphically symplectic manifold is Kähler. **This is false.** D. Guan has constructed examples of manifolds which are compact, simply connected, holomorphically symplectic but non-Kähler. His example was explicated by Bogomolov.

DEFINITION: Let L be a line bundle on an elliptic curve E with the first Chern class $c_1(L) \neq 0$. Denote by \tilde{S} the corresponding \mathbb{C}^* -bundle on E obtained by removing the zero section, $\tilde{S} = \text{Tot}(L) \setminus 0$. Fix a complex number λ with $|\lambda| > 1$, and let $h_\lambda : \tilde{S} \rightarrow \tilde{S}$ be the corresponding homothety of \tilde{S} . The quotient $\tilde{S}/\langle h_\lambda \rangle$ is called **a primary Kodaira-Thurston surface**, or simply **Kodaira-Thurston surface**.

REMARK: Kodaira-Thurston surface is an isotrivial elliptic fibration over the elliptic curve E , with the fiber identified with the elliptic curve $E_L := \mathbb{C}^*/\langle \lambda \rangle$. Therefore, **it is holomorphically symplectic.**

Bogomolov-Guan manifolds

DEFINITION: Let S be a Kodaira-Thurston surface, $S^{[n]}$ its Hilbert scheme, $S^{(n)}$ its symmetric power, and $\pi_S : S \rightarrow E$ the elliptic fibration constructed above. π_S to each component of $S^{(n)}$ and summing up, we obtain a holomorphic projection from $S^{(n)}$ to E ; taking the composition with the resolution $r : S^{[n]} \rightarrow S^{(n)}$, we obtain an isotrivial fibration $\pi : S^{[n]} \rightarrow E$. Denote its fiber by $F^{[n]}$. Then $F^{[n]}$ is a smooth divisor in a holomorphically symplectic manifold $(S^{[n]}, \Omega)$. The restriction of Ω to $F^{[n]}$ has rank $2n - 2$, because $F^{[n]} \subset S^{[n]}$ is a divisor. Denote by $K \subset TF^{[n]}$ the kernel of $\Omega|_{F^{[n]}}$, that is, the set of all $x \in TF^{[n]}$ such that $\Omega|_{F^{[n]}}(x, \cdot) = 0$. The corresponding foliation is called **the characteristic foliation**.

REMARK: The leaf space W of K is a holomorphically symplectic orbifold, but it is never smooth. When the degree of the line bundle L over E is divisible by n , the space W has a smooth finite covering, of order n^2 , ramified in the singular points of W . This covering is called **the Bogomolov-Guan manifold**. By construction, **it is compact, simply connected, holomorphically symplectic**.

REMARK: Since the Bogomolov-Guan manifold contains a blown-up Kodaira-Thurston surface, **it is non-Kähler**.

Bogomolov-Tian-Todorov for holomorphically symplectic manifolds

This is the version of Bogomolov-Tian-Todorov which can be applied to Bogomolov-Guan manifolds.

THEOREM: Let (M, I, Ω) be a compact holomorphically symplectic manifold (not necessarily Kähler). Assume that the Dolbeault cohomology group $H_{\bar{\partial}}^{0,2}(M) = H^2(\mathcal{O}_M)$ is generated by $\bar{\partial}$ -closed $(0, 2)$ -forms, and all $\bar{\partial}$ -exact holomorphic 3-forms on M vanish. **Then the holomorphic symplectic deformations of (M, I, Ω) are unobstructed.** If, in addition, all classes in the Dolbeault cohomology group $H_{\bar{\partial}}^{1,1}(M)$ are represented by closed $(1, 1)$ -forms, **the complex deformations of M are also unobstructed,** and all sufficiently small complex deformations remain holomorphically symplectic.

I would try to explain how such a result can be obtained.

Schouten brackets

DEFINITION: Let M be a complex manifold, and $\Lambda^{0,p}(M) \otimes T^{1,0}M$ the sheaf of $T^{1,0}M$ -valued $(0,p)$ -forms. Consider the commutator bracket $[\cdot, \cdot]$ on $T^{1,0}M$, and let $\bar{\mathcal{O}}_M$ denote the sheaf of antiholomorphic functions. Since $[\cdot, \cdot]$ is $\bar{\mathcal{O}}_M$ -linear, it is naturally extended to $\Lambda^{0,p}(M) \otimes_{C^\infty M} T^{1,0}M = \bar{\Omega}^p M \otimes_{\bar{\mathcal{O}}_M} T^{1,0}M$, giving a bracket

$$[\cdot, \cdot] : \Lambda^{0,p}(M) \otimes T^{1,0}M \times \Lambda^{0,q}(M) \otimes T^{1,0}M \longrightarrow \Lambda^{0,p+q}(M) \otimes T^{1,0}M.$$

This bracket is called **Schouten bracket**.

REMARK: Since $[\cdot, \cdot]$ is $\bar{\mathcal{O}}_M$ -linear, the Schouten bracket satisfies the Leibnitz identity:

$$\bar{\partial}([\alpha, \beta]) = [\bar{\partial}\alpha, \beta] + [\alpha, \bar{\partial}\beta].$$

This allows one to extend the Schouten bracket to the $\bar{\partial}$ -cohomology of the complex $(\Lambda^{0,*}(M) \otimes T^{1,0}M, \bar{\partial})$, which coincide with the cohomology of the sheaf of holomorphic vector fields: $[\cdot, \cdot] : H^p(TM) \times H^q(TM) \longrightarrow H^{p+q}(TM)$.

Tian-Todorov lemma

DEFINITION: Assume that M is a complex n -manifold with trivial canonical bundle K_M , and Φ a non-degenerate section of K_M . We call a pair (M, Φ) a **Calabi-Yau manifold**. Substitution of a vector field into Φ gives an isomorphism $TM \cong \Omega^{n-1}(M)$. Similarly, one obtains an isomorphism

$$\Lambda^{0,q}M \otimes \Lambda^p TM \longrightarrow \Lambda^{0,q}M \otimes \Lambda^{n-p,0}M = \Lambda^{n-q,p}M. \quad (*)$$

Yukawa product $\bullet : \Lambda^{p,q}M \otimes \Lambda^{p_1,q_1}M \longrightarrow \Lambda^{p+p_1-n,q+q_1}M$ is obtained from the usual product

$$\Lambda^{0,q}M \otimes \Lambda^p TM \times \Lambda^{0,q_1}M \otimes \Lambda^{p_1} TM \longrightarrow \Lambda^{0,q+q_1}M \otimes \Lambda^{p+p_1} TM$$

using the isomorphism (*).

TIAN-TODOROV LEMMA: Let (M, Φ) be a Calabi-Yau manifold, and

$$[\cdot, \cdot] : \Lambda^{0,p}(M) \otimes T^{1,0}M \times \Lambda^{0,q}(M) \otimes T^{1,0}M \longrightarrow \Lambda^{0,p+q}(M) \otimes T^{1,0}M.$$

its Schouten bracket. Using the isomorphism (*), we can interpret Schouten bracket as a map

$$[\cdot, \cdot] : \Lambda^{n-1,p}(M) \times \Lambda^{n-1,q}(M) \longrightarrow \Lambda^{n-1,p+q}(M).$$

Then, for any $\alpha \in \Lambda^{n-1,p}(M)$, $\beta \in \Lambda^{n-1,p_1}(M)$, one has

$$[\alpha, \beta] = \partial(\alpha \bullet \beta) - (\partial\alpha) \bullet \beta - (-1)^{n-1+p} \alpha \bullet (\partial\beta),$$

where \bullet denotes the Yukawa product.

Maurer-Cartan equation and deformations

CLAIM: Let (M, I) be an almost complex manifold, and B an abstract vector bundle over \mathbb{C} isomorphic to $\Lambda^{0,1}(M)$. Consider a differential operator $\bar{\partial} : C^\infty M \rightarrow B = \Lambda^{0,1}(M)$ satisfying the Leibnitz rule. Its symbol is a linear map $u : \Lambda^1(M, \mathbb{C}) \rightarrow B$. Then $B = \frac{\Lambda^1(M, \mathbb{C})}{\ker u} = \Lambda^{0,1}(M)$. Extend $\bar{\partial} : C^\infty M \rightarrow B$ to the corresponding exterior algebra using the Leibnitz rule:

$$C^\infty M \xrightarrow{\bar{\partial}} B \xrightarrow{\bar{\partial}} \Lambda^2 B \xrightarrow{\bar{\partial}} \Lambda^3 B \xrightarrow{\bar{\partial}} \dots$$

Then integrability of I is equivalent to $\bar{\partial}^2 = 0$.

Proof: This is essentially the Newlander-Nirenberg theorem. ■

REMARK: Almost complex deformations of I are given by the sections $\gamma \in T^{1,0}M \otimes \Lambda^{0,1}(M)$, with the integrability relation $(\bar{\partial} + \gamma)^2 = 0$ rewritten as **the Maurer-Cartan equation** $\bar{\partial}(\gamma) = -\{\gamma, \gamma\}$. Here $\bar{\partial}(\gamma)$ is identified with the anticommutator $\{\bar{\partial}, \gamma\}$, and $\{\gamma, \gamma\}$ is anticommutator of γ with itself, where γ is considered as a $\Lambda^{0,1}(M)$ -valued differential operator. **This identifies $\{\gamma, \gamma\}$ with the Schouten bracket.**

REMARK: We shall write $[\gamma, \gamma]$ instead of $\{\gamma, \gamma\}$, because this usage is more common.

Maurer-Cartan equation and obstructions to the deformations

The Kuranishi deformation space of complex structures on M **is identified with the space of solutions of Maurer-Cartan equation $\bar{\partial}(\gamma) = -[\gamma, \gamma]$ modulo the diffeomorphism action.**

DEFINITION: Write γ as power series, $\gamma = \sum_{i=0}^{\infty} t^{i+1} \gamma_i$. **Then the Maurer-Cartan becomes**

$$\bar{\partial}\gamma_0 = 0, \quad \bar{\partial}\gamma_p = - \sum_{i+j=p-1} [\gamma_i, \gamma_j]. \quad (**)$$

We say that deformations of complex structures are **unobstructed** if the solutions $\gamma_1, \dots, \gamma_n, \dots$ of (**) can be found for γ_0 in any given cohomology class $[\gamma_0] \in H^1(M, TM)$.

REMARK: Unobstructedness means that **the Kuranishi deformation space K of (M, I) is smooth and the Kodaira-Spencer map $T_{(M, I)}K \rightarrow H^1(M, TM)$ is an isomorphism.**

Tian-Todorov lemma and deformations

REMARK: Notice that the sum $\sum_{i+j=p-1} [\gamma_i, \gamma_j]$ is always $\bar{\partial}$ -closed. Indeed, the Schouten bracket commutes with $\bar{\partial}$, hence

$$\bar{\partial} \left(\sum_{i+j=p-1} [\gamma_i, \gamma_j] \right) = - \sum_{i+j+k=p-1} [\gamma_i, [\gamma_j, \gamma_k]] + [[\gamma_i, \gamma_j], \gamma_k]. \quad (***)$$

vanishes as a sum of triple supercommutators. **Obstructions to deformations** are given by cohomology classes of the sums $\sum_{i+j=p-1} [\gamma_i, \gamma_j]$, which are defined inductively. These classes are called **Massey powers** of γ_0 .

REMARK: In the Kähler setting, **Tian-Todorov lemma immediately implies the unobstructedness of deformations for compact manifolds with trivial canonical bundle.** Indeed, we can always start from $\gamma_0 \in TM \otimes \Lambda^{0,1}(M) = \Lambda^{n-1,1}(M)$ which is harmonic. Then it is $\bar{\partial}$ - and ∂ -closed. Therefore, $[\gamma_0, \gamma_0] = \partial(\gamma_0 \bullet \gamma_0)$ is ∂ -exact. It is also $\bar{\partial}$ -closed, because the Yukawa product commutes with $\bar{\partial}$. Then $\partial\bar{\partial}$ -lemma implies that $[\gamma_0, \gamma_0]$ is $\partial\bar{\partial}$ -exact. Using induction, we may assume that the solutions of (**) for $p = 1, \dots, n-1$ are all $\partial\bar{\partial}$ -exact. To solve (**) for $p = n$, we use Tian-Todorov lemma again, obtaining $\gamma_n = -G_{\bar{\partial}} \left(\sum_{i+j=n-1} \partial(\gamma_i \bullet \gamma_j) \right)$, where $G_{\bar{\partial}}$ is the Green operator inverting $\bar{\partial}$.

Tian-Todorov lemma for holomorphically symplectic manifolds

LEMMA: Let M be a holomorphic symplectic manifold. Consider the operators $L_\Omega(\alpha) := \Omega \wedge \alpha$, H_Ω acting as multiplication by $n - p$ on $\Lambda^{p,q}(M)$, and $\Lambda_\Omega := \star \wedge \star$. **Then $L_\Omega, H_\Omega, \Lambda_\Omega$ satisfy the $\mathfrak{sl}(2)$ relations, similar to the Lefschetz triple: $[H_\Omega, L_\Omega] = 2L_\Omega$, $[H_\Omega, \Lambda_\Omega] = -2\Lambda_\Omega$, $[L_\Omega, \Lambda_\Omega] = H_\Omega$. ■**

Let now Ω be a holomorphically symplectic form on a complex manifold M , $\dim_{\mathbb{C}} M = 2n$. Then $TM \cong \Omega^1 M$, hence the Schouten bracket is defined as

$$\Lambda^{1,p}(M) \times \Lambda^{1,q}(M) \longrightarrow \Lambda^{1,p+q}(M).$$

LEMMA: (Tian-Todorov for holomorphically symplectic manifolds)

Let (M, Ω) be a holomorphically symplectic manifold, and

$$[\cdot, \cdot]_\Omega : \Lambda^{1,p}(M) \times \Lambda^{1,q}(M) \longrightarrow \Lambda^{1,p+q}(M).$$

the Schouten bracket. **Then for any $a, b \in \Lambda^{1,*}(M)$, one has**

$$[a, b] = \delta(a \wedge b) - (\delta a) \wedge b - (-1)^{\tilde{a}} a \wedge \delta(b),$$

where \tilde{a} is parity of a , and $\delta := [\Lambda_\Omega, \partial]$.

Proof: Same as for the usual Tian-Todorov. ■

Maurer-Cartan for Hamiltonian vector fields

REMARK: A solution of the Maurer-Cartan equation $(\bar{\partial} + \sum_{i=0}^{\infty} t^{i+1} \gamma_i)^2 = 0$ gives a holomorphically symplectic deformation whenever all γ_i belong to $\Lambda^{0,1}(M) \otimes \text{Ham}_M$.

Using Ω to identify vector fields and 1-forms, the sheaf of Hamiltonian vector fields can be embedded to $\Lambda^{1,0}(M)$ as a sheaf of ∂ -closed $(1,0)$ -forms.

Similarly, if we use Ω to consider γ_i as sections of $\Lambda^{0,1}(M) \otimes T^{1,0}M = \Lambda^{1,1}(M)$, the condition $\gamma_i \in \Lambda^{0,1}(M) \otimes \text{Ham}_M$ is interpreted as $\partial\gamma_i = 0$.

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold. We say that the holomorphic symplectic deformations of (M, I, Ω) are unobstructed if for any $\bar{\partial}$ - and ∂ -closed $\gamma_0 \in \Lambda^{1,1}(M)$ the Maurer-Cartan equation

$$\bar{\partial}\gamma_p = - \sum_{i+j=p-1} [\gamma_i, \gamma_j], \quad p = 1, 2, 3, \dots$$

has a solution $(\gamma_1, \gamma_2, \dots)$, with $\gamma_i \in \Lambda^{1,1}(M)$ ∂ -closed.

Bogomolov-Tian-Todorov for BG-manifolds

THEOREM 1: Let (M, I, Ω) be a compact holomorphically symplectic manifold (not necessarily Kähler). Assume that the Dolbeault cohomology group $H_{\bar{\partial}}^{0,2}(M) = H^2(\mathcal{O}_M)$ is generated by ∂ -closed $(0,2)$ -forms, and all ∂ -exact holomorphic 3-forms on M vanish. **Then the holomorphic symplectic deformations of (M, I, Ω) are unobstructed.**

Proof: Let $\alpha, \beta \in \Lambda^{1,*}(M)$ be ∂ -closed forms, and $[\alpha, \beta]$ the Schouten bracket. The holomorphic symplectic Tian-Todorov lemma gives $[\alpha, \beta] = \partial \Lambda_{\Omega}(\alpha \wedge \beta)$. Suppose we have solved the Maurer-Cartan equation for all $p < n$, and all $\gamma_p \in \Lambda^{1,1}(M)$ with $p < n$ are ∂ -closed. The Maurer-Cartan for γ_n becomes

$$\bar{\partial} \gamma_n = \sum_{i+j=n-1} \partial \Lambda_{\Omega}(\gamma_i \wedge \gamma_j) \quad (MC)$$

The right hand side of (MC) is $\bar{\partial}$ -closed by the standard argument with triple commutators. Then it is $\bar{\partial}$ -exact by Lemma 1 below, applied to $\rho = \sum_{i+j=n-1} \Lambda_{\Omega}(\gamma_i \wedge \gamma_j)$. ■

LEMMA 1: In assumptions of Theorem 1, let $\rho \in \Lambda^{0,2}(M)$ be a form which satisfies $\bar{\partial} \partial \rho = 0$. **Then $\partial \rho$ is $\partial \bar{\partial}$ -exact.**

Bogomolov-Tian-Todorov for BG-manifolds (2)

LEMMA 1: In assumptions of Theorem 1, let $\rho \in \Lambda^{0,2}(M)$ be a form which satisfies $\bar{\partial}\partial\rho = 0$. **Then $\partial\rho$ is $\partial\bar{\partial}$ -exact.**

Proof: Since $\bar{\partial}\partial\bar{\rho} = 0$, the $(3,0)$ -form $\partial\bar{\rho}$ is ∂ -exact and holomorphic. By assumptions of Theorem 1, it vanishes. Then $\partial\bar{\rho} = 0$. Since all $\bar{\partial}$ -cohomology classes in $\Lambda^{0,2}(M)$ can be represented by closed forms, there exists a $\bar{\partial}$ -exact form $\rho' = \bar{\partial}\mu$ with the same $\partial\rho = \bar{\rho}'$. This gives $\partial\rho = \partial\bar{\partial}\mu$. ■

Antecedent: Hamiltonian vector fields and formal unobstructedness

In a joint work with D. Kaledin, we studied deformation theory of holomorphically symplectic (not necessarily compact). Since then, it was used a lot in the work in symplectic singularities.

Let $\text{Ham}(M)$ be the sheaf of holomorphic Hamiltonian vector fields on M . The corresponding DG-Lie algebra $(\Lambda^{0,*}(M) \otimes \text{Ham}(M), \bar{\partial})$ **is responsible for the holomorphically symplectic deformations of (M, Ω)** . However, the sheaf $\text{Ham}(M)$ can be obtained from

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \mathcal{O}_M \xrightarrow{\Theta} \text{Ham}(M) \longrightarrow 0,$$

\mathbb{C}_M is the constant sheaf, and $\Theta(f) = \Omega^{-1}(df)$ is the Hamiltonian vector field associated with the holomorphic function f . When $H^{>0}(\mathcal{O}_M) = 0$, the multiplication on $H^{>0}(\text{Ham}(M))$ vanishes because $H^i(\text{Ham}(M)) = H^{i+1}(\mathbb{C}_M)$ and the product $H^i(\text{Ham}(M)) \times H^j(\text{Ham}(M)) \longrightarrow H^{i+j}(\text{Ham}(M))$ can be represented as

$$H^i(\text{Ham}(M)) \times H^{j+1}(\mathbb{C}_M) \longrightarrow H^{i+j+1}(\mathbb{C}_M),$$

with the multiplication given by the Poisson product. **The latter clearly vanishes, because the Poisson product with the constant vanishes.**

This way we obtained the following result on deformations.

THEOREM: (Kaledin-V., 2002)

Let M be a holomorphically symplectic algebraic variety, and $H^i(M, \mathbb{C}) \longrightarrow H^i(\mathcal{O}_M)$ is surjective for all $i > 0$. **Then the formal deformations of holomorphic symplectic complex structures on M are unobstructed.**