Bogomolov-Guan precursors and quasi-diagonals in a product of elliptic curves.

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Quasi-diagonals

DEFINITION: Let L be an ample line bundle on an elliptic curve E. Let $p_i: E^n \longrightarrow E$ denotes the projection to i-th component. **A quasidiagonal** is a curve $S \subset E^n$ such that for all i, j, the bundle $V_{i,j} := p_i^* L|_S \otimes (p_j^* L|_S)^{-1}$ is a torsion, that is, satisfies $V_{i,j}^{\otimes k} = \mathcal{O}_S$ for some $k > 0$.

REMARK: The degree of the bundle $p_i^* L|_S$ is equal to $\deg L \cdot d_i$, where d_i is the degree of the restriction $p_i: S \longrightarrow E$. Therefore, **the map $p_i: S \longrightarrow E$ has the same degree for all i,** if S is quasidiagonal.

REMARK: Quasidiagonals **correspond to projective submanifolds in a certain complex non-Kähler holomorphically symplectic manifold,** called “the Bogomolov-Guan manifold”.

THEOREM: **There are at most countably many quasi-diagonals for any given (E, L)** (later today).

Examples of quasidiagonals

REMARK: In the next slide, **we reduce the classification of quasidiagonals in E^n to the classification of quasidiagonals in E^2.**

EXAMPLE: Let $\nu: E \longrightarrow E$ be an automorphism of order d. Then $L := \prod_{i=0}^{d-1} (\nu^i)^* L_1$ is ν-invariant, for any given L_1, hence the graph $\Gamma_\nu \subset E^2$ of ν is a quasi-diagonal for (E, L). Clearly, the action of E on itself by translation is transitive on $\operatorname{Pic}_k(E)$ for any given k. Therefore, **for any bundle of degree $k = ld$ divisible by d, there exists an order d automorphism preserving this bundle**, and **its graph is a quasi-diagonal.**

Quasi-diagonals in E^n and in E^2

REMARK: Let $\Pi: E^n \longrightarrow E^m$ be a projection of E^n to $m \leqslant n$ components. Clearly, $\Pi(S)$ is a quasidiagonal for any quasidiagonal $S \subset E^n$. This implies, in particular,

CLAIM: Consider the projection $p_{i,j}: E^n \longrightarrow E^2$, $p_{i,j}(x_1, ..., x_n) = (x_i, x_j)$. Then, **for any quasi-diagonal $S \subset E^n$, its projection $p_{i,j}(S)$ is a quasi-diagonal in E^2 for any i, j.** Conversely, **$S \subset E^n$ is a quasi-diagonal if $p_{i,j}$ is a quasi-diagonal for any i, j.** ■

Taking fibered products of quasi-diagonals in E^2, we can obtain all quasi-diagonals in E^n.

DEFINITION: Let $f: X \longrightarrow Z$, $g: Y \longrightarrow Z$ be any maps. **The fibered product** $X \times_Z Y$ is the set $\{(x,y) \in X \times Y \mid f(x) = g(y)\}$.

THEOREM: Let $S_1, ..., S_{n-1}$ be quasi-diagonals in E^2. Consider the fibered product $E^n = (E^2) \times_E (E^2) \times_E (E^2) \times_E ... \times_E (E^2)$. where the projections of E^2 to the left factor is $(x,y) \longrightarrow x$ and to the right is $(x,y) \longrightarrow y$. **Then $S_1 \times_E S_2 \times_E ... \times_E S_{n-1}$ is a quasi-diagonal, and all quasi-diagonals are obtained this way.** ■

Restricted quasi-diagonals

DEFINITION: Choosing the unity $e \in E$, we fix the group structure on E. **A restricted quasi-diagonal** in E^n is a quasi-diaonal $C \subset E^n$ which belongs to $\sigma^{-1}(e)$, where $\sigma: E^n \to E$ takes a point $(a_1, ..., a_n)$ to $\sum a_i$.

REMARK: For any elliptic curve E, a line bundle L on E and an integer $n \geqslant 2$ the pair (E, L) **admits a countable set of quasi-diagonals in E^n,** The restricted quasi-diagonals **do not exist for general (E, L).**

EXAMPLE: Let $\nu(x) := -x$ be the automorphism of an elliptic curve. **Then the graph of ν is a resticted quasi-diagonal.** Product of restricted quasi-diagonals is a restricted quasi-diagonal.

EXAMPLE: Let E be an elliptic curve which admits an automorphism of order 3, $E = \mathbb{C}/\mathbb{Z}[\zeta]$ where ζ is a primitive 3-rd degree root of unity. Let S be the image of the following embedding to E^3:

$$\left(\mathrm{id}_E \times \zeta \times \zeta^2\right) : E \to E^3; \qquad x \mapsto (x, \zeta x, \zeta^2 x).$$

Then S is a restricted quasi-diagonal for the line bundle $\mathcal{O}(e)$.

REMARK: Taking fiber products of quasi-diagonals in these two examples, **we can construct a restricted quasi-diagonal in E^n for $E = \mathbb{C}/\mathbb{Z}[\zeta]$ and any integer $n \geqslant 5$.**

Open questions about quasi-diagonals

QUESTION: Can a quasi-diagonal have geometric genus $p_g(C) > 1$? There are examples which are elliptic and singular.

QUESTION: Call a line bundle $L \in \operatorname{Pic}_n(E)$ **commensurable with $\mathcal{O}(n[e])$** if $L \otimes \mathcal{O}(n[e])^{-1}$ is a torsion. **Is it true that there are no restricted quasi-diagonals on (E, L) when L is not commensurable with $\mathcal{O}(n[e])$?**

QUESTION: Consider E^n, where n is odd, and E does not admit a complex multiplication. **Are there any restricted quasi-diagonals in E^n, for some ample line bundle L on E?**

QUESTION: In the case $n = 3$ and $E = \mathbb{C}/\mathbb{Z}[\zeta]$ we have constructed an example of a restricted quasi-diagonal in E^3. **Does E^3 contains any restricted quasi-diagonal except this one?**

Hilbert schemes

REMARK: When M is a complex surface, **the Hilbert scheme of points** $M^{[n]}$ is a natural smooth resolution of the n-th symmetric power of M.

REMARK: If the surface M is holomorphically symplectic, $M^{[n]}$ is also holomorphically symplectic (follows easily from Serre's duality).

REMARK: Hilbert scheme of a K3 surface is a simply connected, holomorphically symplectic manifold. Hilbert scheme of a torus T is not simly connected, but the fiber of its Albanese map $T^{[n]} \longrightarrow T$ has finite fundamental group. The universal cover of this fiber is called **generalized Kummer variety**.

This way one obtains **two main examples of simply connected holomorphically symplectic manifolds.**

REMARK: A. Todorov conjectured that any compact, simply connected holomorphically symplectic manifold is Kähler. D. Guan has constructed examples of manifolds which are compact, simply connected, holomorphically symplectic but non-Kähler. His example was used by Bogomolov to produce what is known as Bogomolov-Guan manifolds.

Kodaira-Thurston surface

DEFINITION: Let L be a line bundle on an elliptic curve E with the first Chern class $c_1(L) \neq 0$. Denote by $\tilde{S}$ the corresponding $\mathbb{C}^*$-bundle on E obtained by removing the zero section, $\tilde{S} = \operatorname{Tot}(L)\backslash 0$. Fix a complex number λ with $|\lambda| > 1$, and let $h_\lambda : \tilde{S} \longrightarrow \tilde{S}$ be the corresponding homothety of $\tilde{S}$. The quotient $\tilde{S}/\langle h_\lambda\rangle$ is called **a primary Kodaira surface**, or **Kodaira-Thurston surface**, or **Kodaira surface**.

REMARK: The Kodaira surface is a principal elliptic fibration over the elliptic curve E, with the fiber identified with the elliptic curve $E_L := \mathbb{C}^*/\langle\lambda\rangle$. Therefore, **it is holomorphically symplectic.**

Bogomolov-Guan manifolds

DEFINITION: Let Kod be a Kodaira surface, $\mathrm{Kod}^{[n]}$ its Hilbert scheme, $\mathrm{Kod}^{(n)}$ its symmetric power, and $\pi: \mathrm{Kod} \longrightarrow E$ the elliptic fibration constructed above. Applying π to each component of $\mathrm{Kod}^{(n)}$ and summing up, we obtain a holomorphic projection from $\mathrm{Kod}^{(n)}$ to E; taking the composition with the resolution $r: \mathrm{Kod}^{[n]} \longrightarrow \mathrm{Kod}^{(n)}$, we obtain an isotrivial fibration $\pi: \mathrm{Kod}^{[n]} \longrightarrow E$. Denote its fiber by F. Then F is a smooth divisor in a holomorphically symplectic manifold $(\mathrm{Kod}^{[n]}, \Omega)$. The restriction of Ω to F has rank $2n-2$, because $F \subset \mathrm{Kod}^{[n]}$ is a divisor. Denote by $K \subset TF$ the kernel of $\Omega|_F$, that is, the set of all $x \in TF$ such that $\Omega|_F(x, \cdot) = 0$. The corresponding foliation is called **the characteristic foliation**.

REMARK: The leaf space W of K is a holomorphically symplectic orbifold, but it is never smooth. When the degree of the line bundle L over E is divisible by n, the space W has a smooth finite covering, of order n^2, ramified in the singular points of W. This covering is called **the Bogomolov-Guan manifold**. By construction, **it is compact, simply connected, holomorphically symplectic.**

Bogomolov-Guan precursor

DEFINITION: Let E_1, E be elliptic curves, and $\text{Kod} \xrightarrow{p} E$ a Kodaira surface, constructed as a principal E_1-bundle over E, and $D \subset \text{Kod}^{n+1}$ the set of all points $(z_1, ..., z_{n+1})$ such that $\sum_{i=1}^{n+1} p(z_i) = 0$. Consider the diagonal E_1-action on D. **The non-restricted Bogomolov-Guan precursor** is the quotient Kod^{n+1}/E_1. **The restricted Bogomolov-Guan precursor** (usually called just "the Bogomolov-Guan precursor") is D/E_1.

REMARK: Let Z be a divisor in a holomorphically symplectic manifold (M, Ω). Then the foliation $\ker \Omega|_Z$ is called **the characteristic foliation** of Z, and its leaf space is holomorphically symplectic.

PROPOSITION: **The orbits is E_1-action on D are leaves of the characteristic foliation**.

COROLLARY: The Bogomolov-Guan precursor X **is holomorphically symplectic,** and the natural map $P: X \longrightarrow B$ to the divisor $B := \{(z_1, ..., z_{n+1}) \in E^{n+1} \mid \sum z_i = 0\}$ is a Lagrangian fibration. ■

THEOREM: A finite quotient of the Bogomolov-Guan manifold **is bimeromorphic to a finite quotient of the Bogomolov-Guan precursor.**

Proof: Clear from the construction.

Projective subvarieties in Bogomolov-Guan precursors

THEOREM: Let $P: X \longrightarrow B$ be the Lagrangian fibration on the Bogomolov-Guan precursor, and $Z \subset X$ an irreducible subvariety. **Then Z is projective if and only if $P(Z) \subset B \subset E^{n+1}$ is a point or a quasi-diagonal.**

Idea of the proof: The projection $X \to B$ is a principal toric fibration, obtained by taking a collection of line bundles $L_1, ..., L_n$ on $B = E^n$ and taking the quotient of the corresponding principal $(\mathbb{C}^*)^n$-bundle by $\mathbb{Z}^n$. If such a fibration has a section, it is trivial. If it has a multisection, a certain tensor power of these line bundles is trivial.

REMARK: Note that **all quasi-diagonals in $B \subset E^{n+1}$ are restricted.** The same argument proves the following

THEOREM: Let $M \to E^n$ be the unrestricted Bogomolov-Guan precursor, and and $Z \subset X$ an irreducible subvariety. **Then Z is projective if and only if $P(Z) \subset E^{n+1}$ is a point or a quasi-diagonal.**

Nilmanifolds

The Bogomolov-Guan precursor is easy to study because **it is a complex nilmanifold.**

DEFINITION: Let M be a smooth manifold equipped with a transitive action of a nilpotent Lie group. Then M is called **a nilmanifold**.

REMARK: All nilmanifolds are obtained as quotient spaces, $M = G/H$.

THEOREM: (Malčev)
Let $\mathfrak{g}$ be a nilpotent Lie algebra defined over $\mathbb{Q}$, and G its Lie group. **Then G contains a discrete subgroup Γ such that G/Γ is compact,** and $\Gamma = e^{\Gamma_\mathfrak{g}}$, where $\Gamma_\mathfrak{g}$ is a lattice subalgebra in $\mathfrak{g}$. Moreover, $\mathfrak{g} \cong \Gamma_\mathfrak{g} \otimes_\mathbb{Q} \mathbb{R}$. Finally, **all nilmanifolds are obtained this way.**

REMARK: Topologically, **all simply connected nilpotent Lie groups are diffeomorphic to $\mathbb{R}^n$,** and all nilmanifolds are **iterated circle fibrations**.

Complex nilmanifolds

DEFINITION: **An integrable complex structure** on a real Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ such that $\mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$

REMARK: Any such decomposition defines a complex structure I on $\mathfrak{g}$ by $I\Big|_{\mathfrak{g}^{1,0}} = \sqrt{-1}$ and $I\Big|_{\mathfrak{g}^{0,1}} = -\sqrt{-1}$. We extend it to a left-invariant almost complex structure on the corresponding Lie group G. Integrability of this complex structure is given by $[T^{1,0}G, T^{1,0}G] \subset T^{1,0}G$, which is equivalent to $[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0}$.

REMARK: Left-invariant complex structures on a connected real Lie group **are in 1 to 1 correspondence with integrable complex structures** on its Lie algebra.

DEFINITION: A **complex nilmanifold** is a nilmanifold $M = G/\Gamma$ equipped with a complex structure, in such a way that the projection $(G, I) \longrightarrow M$ is holomorphic, where I is a left invariant complex structure on G.

REMARK: **This construction would work for any Lie group,** giving left-invariant complex structures on G and its left quotients by any discrete subgroup.

Kodaira surface as a complex nilmanifold

Let $G := \mathbb{R} \times G_0$, where G_0 is the 3-dimensional real Lie group of upper triangular matrices 3x3. This group contains many cocompact lattices, for example $\Gamma := \mathbb{Z} \times \Gamma_0$, where Γ_0 of integer matrices. The corresponding Lie algebra $\mathfrak{g}$. **is generated by x, y, z, t with the only non-zero commutator $[x, y] = z$.**

DEFINITION: The Kodaira surface can be defined as $M := G/\Gamma$ with the complex structure defined by the subalgebra $\mathfrak{g}^{1,0} := \langle x+\sqrt{-1}y, z+\sqrt{-1}t\rangle$, which is actually abelian.

REMARK: From the construction of the complex structure on M, **it is apparent that this projection is holomorphic**; its fibers and its base **is 2-dimensional nilmanifolds, that is, elliptic curves. This implies that G/Γ is a primary Kodaira surface in the sense of the definition given above.**

REMARK: For a complete classification of 2-dimensional complex nilmanifolds and solvmanifolds, see *Keizo Hasegawa, Complex and Kahler structures on Compact Solvmanifolds, J. Symplectic Geom. Volume 3, Number 4 (2005), 749-767*, `https://arxiv.org/abs/0804.4223`.

Barlet spaces

DEFINITION: Barlet spaces are spaces of cycles, that is, closed complex analytic subvarieties of a given dimension in a given complex manifold with multiplicities (positive integers) assigned to their irreducible components. They are similar but distinct from the **Douady spaces,** that are spaces of closed complex analytic subspaces (possibly with nilpotents in the structure sheaf).

DEFINITION: Let M be a metric space.Recall from that the Hausdorff metric on the set $\mathcal{C}$ of closed subsets of M is defined as follows: $d(X,Y)$ is the infimum of all ε such that X belongs to an ε-neighbourhood of Y, and Y belongs to an ε-neighbourhood of X. When M is compact, the corresponding topology on $\mathcal{C}$ is independent on the choice of metric on M as long as the topology of M remains the same. It is called **the Hausdorff topology**.
REMARK: The Barlet space of cycles is complex analytic, with the topology that is **induced by the Hausdorff topology on the set of all closed subvarieties.**

THEOREM: (Bishop)
Let (M,I) be a compact complex manifold, and ω a Hermitian form. Fix $a\in\mathbb{R}$ and let Z_a of the Barlet space $\mathfrak{B}_k(M)$ of k-cycles $Z\subset M$ such that $\int_Z \omega^k \leqslant a$. **Then Z_a is compact.**

SKT manifolds and Barlet spaces

DEFINITION: A Hermitian form ω on a complex manifold is **SKT** (strong Kähler torsion) or **pluriclosed** if $dd^c\omega = 0$.

THEOREM: (Bishop-Gromov)
Let M be a compact Hermitian almost complex manifold, $\mathfrak{X}$ the space of all complex curves on M, and $\mathfrak{X} \xrightarrow{\text{Vol}} \mathbb{R}^{>0}$ the volume function. Then Vol is **proper** (preimage of a compact set is compact).

COROLLARY: Let M be a complex manifold, equipped with a pluriclosed Hermitian form ω, and X a component of the moduli of complex curves. **Then the function $\text{Vol}: X \longrightarrow \mathbb{R}^{>0}$ is constant, and X is compact.**

Proof: Since $\text{Vol} \geqslant 0$, the set $\text{Vol}^{-1}(]-\infty, C])$ is compact for all $C \in \mathbb{R}$, hence Vol has a minimum somewhere in X. However, **a pluriharmonic function which has a minimum is necessarily constant** (E. Hopf's strong maximum principle). Therefore, Vol is constant: $\text{Vol} = A$. Now, compactness of $X = \text{Vol}^{-1}(A)$ follows from Gromov's theorem. ■

THEOREM: The Bogomolov-Guan precursor is SKT.

Proof: Next slide.

SKT metric on the Bogomolov-Guan precursor

Let $\omega = a\wedge b + c\wedge d$ the Gauduchon metric on Kod, written in terms of the frame associated with its Lie algebra as above, and a_i, b_i, c_i, d_i, $i = 1, ..., n+1$ the corresponding 1-forms on Kod^{n+1}. Consider the Hermitian form $\omega := \sum_i a_i \wedge b_i + c_i \wedge d_i$ on Kod^{n+1}. It is not hard to see that $dd^c\omega = 0$.

REMARK: Let $X_1 := \mathrm{Kod}^{n+1}/E_1$, where E_1 acts diagonally, and $\mathrm{Kod}^{n+1} \xrightarrow{\pi} X_1$ the corresponding holomorphic projection. Denote by

$$\pi_*: \Lambda^{p,q}(\mathrm{Kod}^{n+1}) \longrightarrow \Lambda^{p-1,q-1}(X_1)$$

the pushforward, that is, the fiberwise integration of differential forms. Clearly, $\pi_*(\omega^2)$ is a Hermiian form on X_1. Since π_* commutes with d, d^c, to show that $dd^c\pi_*(\omega^2) = 0$ it would suffice to show that

$$dd^c\pi_*(\omega^2) = \pi_*(dd^c\omega\wedge\omega) + \pi_*(d\omega\wedge d^c\omega) = \pi_*(d\omega\wedge d^c\omega)$$

vanishes.

The following proposition immediately implies that X_1, and hence $X \subset X_1$ is an SKT manifold.

SKT metric on the Bogomolov-Guan precursor (2)

PROPOSITION: Let $\operatorname{Kod}^{n+1} \xrightarrow{\pi} X_1$ be the holomorphic projection and ω be the SKT Hermitian form defined above. **Then** $\pi_*(d\omega \wedge d^c\omega) = 0$.

Proof: For any invariant differential form, expressed as a Grassmann polynomial of a_i, b_i, c_i, d_i, the fiberwise integration is given by the operator $\sum_{i=1}^{n+1} i_{z_i} i_{t_i}$, where i denotes the convolution with a vector field, and z_i, t_i the generators of the vector fields tangent to the components of Kod^{n+1} associated with the vector fields $x, y, z, t \in T\operatorname{Kod}$. Clearly, $d\omega = \sum_{i=1}^{n+1} a_i \wedge b_i \wedge d_i$, and $d^c\omega = -\sum_{i=1}^{n+1} a_i \wedge b_i \wedge c_i$, hence $d\omega \wedge d^c\omega = \sum_{i<j} a_i \wedge b_i \wedge a_j \wedge b_j \wedge c_j \wedge d_i$. After contracting with the bivector $z_i \wedge t_i$, this form vanishes. ■

Campana's coreduction map

DEFINITION: A proper holomorphic map $f: X \longrightarrow Y$ of complex varieties is called **projective** if there exist a line bundle L on X which is ample on all fibers of f. It is called **Moishezon** if there exists a bimeromorphism $\mu: X \longrightarrow \tilde X$ and a projective morphism $\tilde f: \tilde X \longrightarrow Y$ such that $f = \mu \circ \tilde f$.

THEOREM: Let $f: X \longrightarrow Y$ be a surjective holomorphic map of irreducible compact complex varieties with Moishezon fibers. Assume that all components of the Barlet space of curves on X are compact, and there exists a subvariety $A \subset X$ such that $f(A) = Y$ and $f|_A$ is Moishezon. **Then X is Moishezon.**

Proof: *Campana F. Coreduction algebraique d'un espace analytique failebement Kahlerian compact// Invent, math. - 1981. - 63. - pp 187-223* ■

This leads to the following theorem

THEOREM: Let X be a compact complex variety, such that all components of the Barlet space of curves on X are compact. Then there exists a meromorphic map $f: X \longrightarrow Y$ with Moishezon fibers, called **the coreduction** of X, such that **for a general point $x \in X$ any Moishezon subvariety passing through x lies in the fiber of f.**

Proof: *Campana F. Coreduction algebraique d'un espace analytique failebement Kahlerian compact// Invent, math. - 1981. - 63. - pp 187-223* ■

Countability of quasi-diagonals

THEOREM: There is at most countably many quasidiagonals for any given (E, L).

Proof. Step 1: Since quasi-diagonals in E^n are obtained from quasi-diagonals in E^2, it suffices to show that there are at most countably many quasi-diagonals in E^2. If the number of quasi-diagonals is incountable, there are continuous families of quasi-diagonals, which cover E^2. **We make this assumption and show that it leads to contradiction.**

Step 2: Denote by B a positive-dimensional irreducible component of the Barlet space of quasi-diagonals, and let C_t, $t \in B$ be the quasi-diagonals. Let $P: \ X \longrightarrow E^2$ be the non-restricted Bogomolov-Guan precursor associated with L. Since $P_d^{-1}(C_t) \xrightarrow{P_d} C_t$ is a projective E_1-fibration, it admits a family of multisections parametrized by E_1 and passing through every point of X.

Step 3: Using the curves constructed in Step 2, we obtain **a collection of curves connecting any two points in X.** Using the coreduction theorem of Campana, this implies that X is Moishezon, a contradiction: if a nilmanifold is Moishezon, it is a torus. ■