Basic facts about hyperkahler and holomorphically symplectic manifolds

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Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

REMARK: The commutator defines a $\mathbb{C}^\infty M$-linear map $N := \Lambda^2(T^{1,0}) \longrightarrow T^{0,1}M$, called **the Nijenhuis tensor** of I. **One can represent N as a section of $\Lambda^{2,0}(M) \otimes T^{0,1}M$.**

Exercise: Prove that $\mathbb{C}P^n$ **is a complex manifold,** in the sense of the above definition.

Kähler manifolds

DEFINITION: A Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Examples of Kähler manifolds.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n = \mathbb{C}^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: Fubini-Study form is Kähler. Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish, because $-\mathrm{Id} \in U(n)$

REMARK: The same argument works for all symmetric spaces.

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler. Indeed, a restriction of a closed form is again closed.

Connections

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: A **connection** on a vector bundle B is a map $B \xrightarrow{\nabla} \Lambda^1 M \otimes B$ which satisfies

$$\nabla(fb) = df \otimes b + f\nabla b$$

for all $b \in B$, $f \in C^\infty M$.

REMARK: A connection ∇ on B gives a connection $B^* \xrightarrow{\nabla^*} \Lambda^1 M \otimes B^*$ on the dual bundle, by the formula

$$d(\langle b, \beta\rangle) = \langle \nabla b, \beta\rangle + \langle b, \nabla^* \beta\rangle$$

These connections are usually denoted **by the same letter ∇.**

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

Levi-Civita connection

DEFINITION: **Torsion** of a connection ∇ is $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$, where $X, Y \in TM$.

An exercise: Prove that torsion is a $C^\infty M$-linear.

DEFINITION: Let (M,g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g) = 0$. It is called **Levi-Civita** if it is torsion-free.

THEOREM: ("the main theorem of differential geometry")
For any Riemannian manifold, the Levi-Civita connection exists, and it is unique.

Levi-Civita connection and Kähler geometry

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) **The complex structure I is integrable, and the Hermitian form ω is closed.**

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection.

REMARK: The implication (ii) $\Rightarrow$ (i) is clear. Indeed, $[X, Y] = \nabla_X Y - \nabla_Y X$, hence it is a (1,0)-vector field when X, Y are of type (1,0), and then I **is integrable**. Also, $d\omega = 0$, **because ∇ is torsion-free,** and $d\omega = \operatorname{Alt}(\nabla\omega)$.

The implication (i) $\Rightarrow$ (ii) is proven by the same argument as used to construct the Levi-Civita connection.

Holonomy group

DEFINITION: (Cartan, 1923) Let (B, ∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B, ∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: A bundle is **flat** (has vanishing curvature) **if and only if its restricted holonomy vanishes.**

REMARK: If $\nabla(\varphi) = 0$ for some tensor $\varphi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$, **the holonomy group preserves φ.**

DEFINITION: Holonomy of a Riemannian manifold is holonomy of its Levi-Civita connection.

EXAMPLE: Holonomy of a Riemannian manifold lies in $O(T_xM, g|_x) = O(n)$.

EXAMPLE: Holonomy of a Kähler manifold lies in $U(T_xM, g|_x, I|_x) = U(n)$.

REMARK: The holonomy group **does not depend on the choice of a point $x \in M$.**

Ambrose-Singer theorem

DEFINITION: Let (B, ∇) be a bundle with connection, $\Theta \in \Lambda^2(M) \otimes \operatorname{End}(B)$ its curvature, and $a, b \in T_xM$ tangent vectors. An endomorphism $\Theta(a, b) \in \operatorname{End}(B)|_x$ is called **a curvature element**.

THEOREM: (Ambrose-Singer) The restricted holonomy group of B, ∇ at $z \in M$ is a Lie group, **with its Lie algebra generated by all curvature elements $\Theta(a, b) \in \operatorname{End}(B)|_x$ transported to z along all paths.**

Holonomy representation

DEFINITION: Let (M, g) be a Riemannian manifold, G its holonomy group. A **holonomy representation** is the natural action of G on TM.

THEOREM: (de Rham) Suppose that the holonomy representation is not irreducible: $T_xM = V_1 \oplus V_2$. Then M locally splits as $M = M_1 \times M_2$, with $V_1 = TM_1$, $V_2 = TM_2$.

Proof. Step 1: Using the parallel transform, we extend $V_1 \oplus V_2$ to a **splitting of vector bundles $TM = B_1 \oplus B_2$, preserved by holonomy.**

Step 2: The sub-bundles B_1, $B_2 \subset TM$ **are integrable:** $[B_1, B_1] \subset B_i$ (the Levi-Civita connection is torsion-free)

Step 3: Taking the leaves of these integrable distributions, **we obtain a local decomposition $M = M_1 \times M_2$, with $V_1 = TM_1$, $V_2 = TM_2$.**

Step 4: Since the splitting $TM = B_1 \oplus B_2$ is preserved by the connection, **the leaves M_1, M_2 are totally geodesic.**

Step 5: Therefore, **locally M splits (as a Riemannian manifold)**:
$M = M_1 \times M_2$, where M_1, M_2 are any leaves of these foliations. ■

The de Rham splitting theorem

COROLLARY: Let M be a Riemannian manifold, and $\mathcal{H}\mathrm{ol}_0(M) \xrightarrow{\rho} \mathrm{End}(T_xM)$ a reduced holonomy representation. Suppose that ρ is reducible: $T_xM = V_1 \oplus V_2 \oplus ... \oplus V_k$. **Then $G = \mathcal{H}\mathrm{ol}_0(M)$ also splits: $G = G_1 \times G_2 \times ... \times G_k$,** with each G_i acting trivially on all V_j with $j \neq i$.

Proof: Locally, this statement follows from the local splitting of M proven above. To obtain it globally in M, use the Lasso Lemma. ■

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

REMARK: It is easy to find non-complete or non-simply connected counterexamples to de Rham theorem.

Simons' theorem

DEFINITION: A symmetric space is a complete Riemannian manifold X such that for all $x \in X$ there exists an isometry of X fixing x and acting as -1 in T_xX.

EXERCISE: Prove that **isometry group acts transitively on any symmetric manifold.**

THEOREM: (Simons, 1962) Let M be a manifold with irreducible holonomy. **Then either M is locally symmetric, or $\mathcal{H}ol(M)$ acts transitively on the unit sphere in T_xM.**

James Harris Simons, 1938-2024

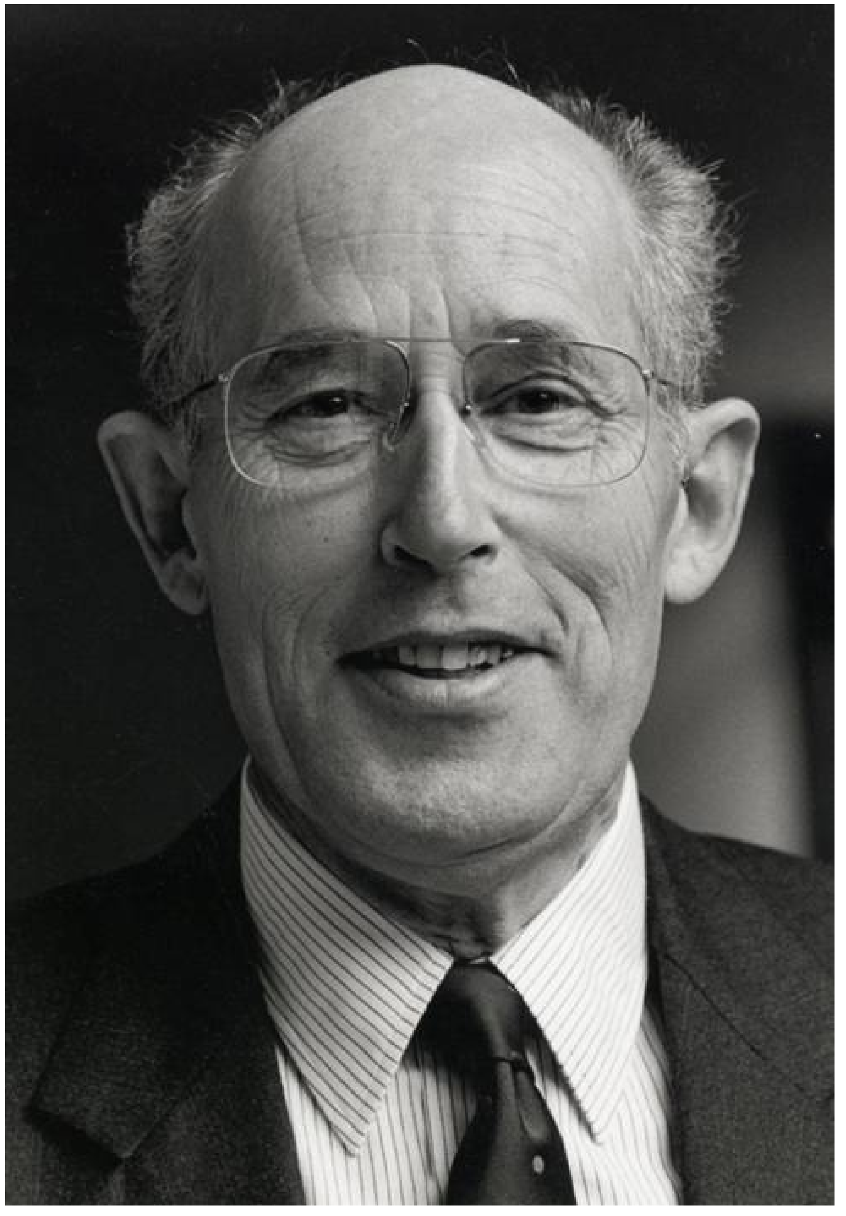
Berger's theorem

THEOREM: (Berger's theorem, 1955) Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

REMARK: There is one more group acting transitively on a sphere: $Spin(9)$ acting on $S^{15} \subset \mathbb{R}^{16}$. In 1968, D. Alekseevsky has shown that **a manifold with holonomy $Spin(9)$ is always locally symmetric.**

REMARK: A similar list exists for non-orthogonal irreducible holonomy without torsion (Merkulov, Schwachhöfer, 1999).



Marcel Berger (1927 - 2016)

Hyperkähler manifolds

REMARK: A Riemannian manifold **is Kähler if and only if the holonomy of its Levi-Civita connection belongs to $U(n)$.**

DEFINITION: Let $V = \mathbb{R}^{4n} = \mathbb{H}^n$ be a quaternionic vector space. **Quaternionic Hermitian form** is a Eucidean metric h on V which is invariant under the action of I, J, K. A **unitary quaternionic map** is an $\mathbb{H}$-linear map $V \longrightarrow V$ which preserves the metric.

DEFINITION: $\mathrm{Sp}(n) = U(n, \mathbb{H})$ is the group of unitary quaternionic matrices.

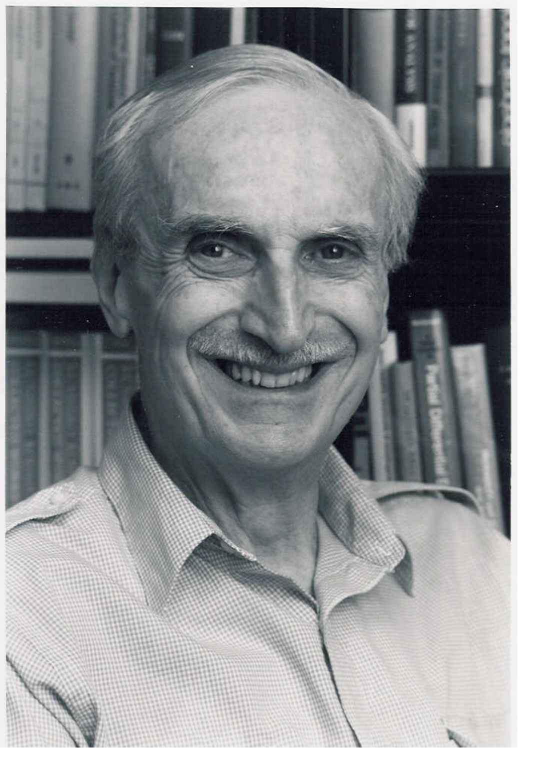
DEFINITION: A hyperkähler manifold is a Riemannian manifold such that the holonomy of its Levi-Civita connection belongs to $Sp(n)$

Hyperkähler manifolds (2)

DEFINITION: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators $I,J,K : TM \longrightarrow TM$, satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are Kähler. Then (M,I,J,K,g) is called **hyperkähler**.



Eugenio Calabi, b. 1923

REMARK: This is the same as $\mathcal{H}\mathrm{ol}(M) \subset \mathrm{Sp}(n)$.
Indeed, if $\mathcal{H}\mathrm{ol}(M) \subset \mathrm{Sp}(n)$, we have 3 complex structures $I,J,K : TM \longrightarrow TM$, such that $\nabla(I) = \nabla(J) = \nabla(K) = 0$, which implies that I,J,K are Kähler. Conversely, if I,J,K are Kähler, we have $\nabla(I) = \nabla(J) = \nabla(K) = 0$.

Holomorphic symplectic geometry

REMARK: A hyperkähler manifold (M, I, J, K) is equipped with 3 symplectic forms ω_I, ω_J, ω_K, with

$$\omega_I(x,y) := g(x, Iy),\ \omega_J(x,y) := g(x, Jy),\ \omega_K(x,y) := g(x, Ky).$$

LEMMA: The form $\Omega := \omega_J + \sqrt{-1}\omega_K$ **is a holomorphic symplectic 2-form on** (M, I)**.** ■

Converse is also true, as follows from the famous conjecture, made by Calabi in 1952.

THEOREM: (S.-T. Yau, 1978)
Let M be a compact, holomorphically symplectic Kähler manifold. Then M **admits a hyperkähler metric,** which is uniquely determined by the cohomology class of its Kähler form ω_I.

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

Eugenio Calabi (1923-2023), Shing-Tung Yau (b. 04.04.1949)

Bogomolov's decomposition theorem

THEOREM: (Cheeger-Gromoll) Let M be a compact Ricci-flat Riemannian manifold with $\pi_1(M)$ infinite. **Then a universal covering of M is a product of $\mathbb{R}$ and a Ricci-flat manifold.**

COROLLARY: A fundamental group of a compact Ricci-flat Riemannian manifold is **"virtually polycyclic": it has a finite index free abelian subgroup.**

REMARK: This is equivalent to any compact Ricci-flat manifold having a finite covering which has free abelian fundamental group.

REMARK: This statement contains the Bieberbach's solution of Hilbert's 18-th problem on classification of crystallographic groups.

THEOREM: (Bogomolov's decomposition) Let M be a compact, Ricci-flat Kaehler manifold. **Then there exists a finite covering $\tilde M$ of M which is a product of Kaehler manifolds of the following form:**

$$\tilde M = T \times M_1 \times ... \times M_i \times K_1 \times ... \times K_j,$$

with all M_i, K_i simply connected, T a torus, and $\mathcal{H}ol(M_l) = Sp(n_l)$, $\mathcal{H}ol(K_l) = SU(m_l)$

The Bogomolov-Beauville-Fujiki (BBF) form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$,** and $c > 0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki (BBF) form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ -\frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(3, b_2 - 3)$.** It is negative definite on primitive forms, and positive definite on $\langle\Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

COROLLARY: The space $H^{1,1}(M)$ of I-invariant cohomology classes has signature $(1, b_2 - 2)$ (**hyperbolic signature**).