Multiplicity of singularities is not a bi-Lipschitz invariant

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Zariski conjecture

PROPOSITION: Let Z be a germ of a complex hypersurface in $\mathbb{C}^n$. **Then a generic linear projection $\pi: Z \longrightarrow \mathbb{C}^{n-1}$ is a finite map** (that is, Z is a graph of a multi-valued function on $\mathbb{C}^{n-1}$.)

DEFINITION: Degree of this map (number of preimages) is called **multiplicity of the singularity** $z \in Z$.

REMARK: It is independent from the complex embedding $Z \subset \mathbb{C}^n$. The same way **one defines multiplicity for any singularity.**

Zariski multiplicity conjecture (1971): Let $Z_1, Z_2 \subset \mathbb{C}^n$ be germs of hypersurfaces in 0, and $\Phi: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ a germ of a homeomorphism inducing a homeomorphism between Z_1 and Z_2. **Then Z_1, Z_2 have the same multiplicity.**

REMARK: Zariski has proven this conjecture when $\dim Z = 1$**.** For any other dimension **Zariski conjecture is open.**

What is known about Zariski conjecture. Bi-Lipschitz maps.

(Campo, Lê, 1973): Zariski conjecture is true **if Z_1 is smooth**

Zariski conjecture **is known for** $\dim Z_i = 2$ **and:**

1. multiplicity 2 (Navarro Aznar, 1980)

2. isolated, homogeneus singularities (Xu-Yau, 1989)

DEFINITION: A map $f: M_1 \longrightarrow M_2$ between metric spaces is **bi-Lipschitz** if there is a constant $C > 1$ such that for any $x, y \in M_1$, one has

$$C^{-1} d(x,y) \leqslant d(f(x), f(y)) \leqslant C d(x,y).$$

Then C is called **(bi-)Lipschitz constant.**

DEFINITION: Two germs $Z, Z' \subset \mathbb{C}^n$ are called **ambient bi-Lipschitz equivalent** if they are homeomorphic, and this homeomorphism is induced by a bi-Lipschitz map on $\mathbb{C}^n$.

THEOREM: **(Risler-Trotman, 1980) Anbient bi-Lipschitz equivalent singularities have the same multiplicities.**

Generalizations of Risler-Trotman

What about other dimensions?

THEOREM: (Comte, 1998) Let $Z_1, Z_2 \subset \mathbb{C}^n$ be germs of k-dimensional subvarieties. Then there exists a constant ε, depending only on k and n, such that **any germs $Z_1, Z_2 \subset \mathbb{C}^n$ which are ambient bi-Lipschitz invariant with Lipschitz constant $C < 1 + \varepsilon$ have the same multiplicity.**

QUESTION: Can we drop the $C < 1 + \varepsilon$ restriction in this theorem?

THEOREM: (**L. Birbrair, A. Fernandes, D. T. Lê and J. E. Sampaio)** Let $Z \subset \mathbb{C}^n$ be a germ subvariety which is ambient bi-Lipschitz equivalent to smooth. **Then Z is smooth.**

THEOREM: (J. F. de Bobadilla, A. Fernandes and J. E. Sampaio) Let $Z_1, Z_2 \subset \mathbb{C}^n$ be germs of k-dimensional subvarieties of dimension 1 or 2, which are ambient bi-Lipschitz invariant. **Then they have the same multiplicity.**

THEOREM: (L. Birbrair, A. Fernandes, J. E. Sampaio, V.) There are **ambient bi-Lipschitz equivalent subvarieties of dimension 3 with isolated singularities which have non-equal multiplicities.**

Zariski multiplicity conjecture for families

CONJECTURE: (Zariski multiplicity conjecture for families; still open) Let X_t be a family of isolated singularities of hypersurfaces which is **topologically trivial** (that is, the group of homeomorphisms of the family acts transitively on fibers). **Then the multiplicities of X_t are constant**.

DEFINITION: Let X be a germ of hypersurface in $\mathbb{C}^n$ defined by an analytic equation $f = 0$, where f has no multiple factors. **Milnor number** of X is $\dim \mathbb{C}[[x_1, ..., x_n]]/\langle\frac{\partial f}{\partial x_i}\rangle$. **A μ-constant family** is a family of isolated singularities with constant Milnor number.
THEOREM: (Milnor, 1968)
Any topologically trivial family is μ-constant.

THEOREM: If $n \neq 3$, **any μ-constant family is topologically trivial.**
Proof: Lê-Ramanujam (1997); for $n = 3$ this is still open. ■

THEOREM: (J. F. de Bobadilla, T. Pełka, 2021)
Any μ-constant family has constant multiplicity.

REMARK: This implies the Zariski conjecture for families.

REMARK: Its proof **is based on MacLean's algebraic-geometric interpretatuon of Floer cohomology.**

Homogeneous singularities

DEFINITION: Let $Z \subset \mathbb{C}P^n$ be a projective variety, given by a homogeneous polynomials $P_1, ..., P_k \in \mathbb{C}[z_1, ..., z_{n+1}]$ The set $C(Z)$ of common zeros of $P_1, ..., P_k$ in $\mathbb{C}^{n+1}$ is called **projective cone** of Z. Clearly, **$C(Z)\backslash\{0\}$ is fibered over Z with fiber $\mathbb{C}^n$.**

DEFINITION: **Homogeneous singularity** is a singularity of a projective cone.

PROPOSITION: Let $X \subset \mathbb{C}P^n$ be a projective variety, $x \in X$ a point with homogeneous singularity, and $Z \subset \mathbb{C}P^{n-1}$ its projectivized tangent cone. **Then the multiplicity of X in x is equal to the degree of Z.**

Strategy of finding bi-Lipschitz equivalent germs of different degree: find two projective varieties X, Y of different degree such that their cones are diffeomorphic.

THEOREM: There exists two 2-dimensional projective varieties $X_1, X_2 \subset \mathbb{C}P^n$, both biholomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, **such that the cones $C(X_1) \subset \mathbb{C}^{n+1}$ and $C(X_2) \subset \mathbb{C}^{n+1}$ are bi-Lipschitz equivalent, but X_1, X_2 have different degree.**

Linking form

CLAIM: Let M be a compact manifold. Denote by $\tau H^{n-i}(M)$ the torsion part of cohomology. **Then**

$$\tau H^{n-i}(M) = \operatorname{Ext}(H_{n-i-1}(M), \mathbb{Z}) = \operatorname{Hom}(\tau H_{n-i-1}(M), \mathbb{Q}/\mathbb{Z}).$$

DEFINITION: Let M be an odd-dimensional manifold, $\dim M = 2k+1$. Define **the linking form**

$$\tau H_k(M) \otimes \tau H_k(M) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

using the isomorphism $\tau H^{k+1}M = \operatorname{Hom}(\tau H_k M, \mathbb{Q}/\mathbb{Z})$ and the Poincaré isomorphism $H_k(M) = H^{k+1}M$.

REMARK: If M is rational homology sphere, the linking form σ is symmetric when n is odd, and antisymmetric otherwise. **For a 5-manifold M, σ is antisymmetric when M is spin.**

Classification of 5-manifolds

DEFINITION: A simply connected, compact, oriented 5-manifold is called **Smale-Barden manifold**.

The Smale-Barden manifolds are **uniquely determined by their second Stiefel-Whitney class and the linking form.**

THEOREM: Let X, X' be two Smale-Barden manifolds. Assume that $H^2(X) = H^2(X')$ and this isomorphism is compatible with the linking form and preserves the second Stiefel-Whitney class. **Then X is diffeomorphic to X'.**

COROLLARY: There exists only two Smale-Barden manifolds M with $H^2(M) = \mathbb{Z}$: the product $S^2 \times S^3$ and the total space of a non-trivial S^3-bundle over S^2.

Proof: Indeed, the linking form on $\mathbb{Z}$ vanishes, therefore the manifold is uniquely determined by the Stiefel-Whitney class $w_2(M)$. Hence we have only two possibilities: $w_2(M) = 0$ and $w_2(M) \neq 0$. ■

S^1-fibrations over projective manifolds

Proposition 1: Let $\pi : M \longrightarrow B$ be a simply connected 5-manifold obtained as a total space of an S^1-bundle L over $B = \mathbb{C}P^1 \times \mathbb{C}P^1$. **Then $H^2(M)$ is torsion-free, and M is diffeomorphic to $S^2 \times S^3$.**

Proof. Step 1: Universal coefficients formula gives an exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H_1(M;\mathbb{Z}),\mathbb{Z}) \longrightarrow H^2(M;\mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_2(M;\mathbb{Z}),\mathbb{Z}) \longrightarrow 0.$$

This implies that $H^2(M;\mathbb{Z})$ is torsion-free.

Step 2: Consider the following exact sequence of homotopy groups

$$0 \longrightarrow \pi_2(M) \longrightarrow \pi_2(B) \xrightarrow{\varphi} \pi_1(S^1) \longrightarrow \pi_1(M) = 0$$

Since $\pi_1(M) = 0$, the map φ, representing the first Chern class of L, is surjective. This exact sequence becomes

$$0 \longrightarrow \pi_2(M) \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0$$

giving $\pi_2(M) = \mathbb{Z}$, and $H^2(M) = \mathbb{Z}$ because $H^2(M)$ is torsion-free.

Step 3: To deduce Proposition 1 from the Smale-Barden classification, it remains to show that $w_2(M) = 0$. **However, $w_2(M) = \pi^*(w_2(B))$** and the latter vanishes, because $w_2(S^2) = 0$. ■

Links of projective cones of $\mathbb{C}P^1 \times \mathbb{C}P^1$

Proposition 2: Let $X \subset \mathbb{C}P^n$ be a variety isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, and $S := C(X) \cap S^{2n+1}$ its link. Assume that and $\mathcal{O}(1)|_X = \mathcal{O}(a,b)$. **Then X has degree $2ab$.** If, in addition, a and b are relatively prime, **the link of $C(X)$ is diffeomorphic to $S^2 \times S^3$.**

Proof. Step 1: Clearly, $c_1(\mathcal{O}(a,b))^2 = 2ab$. On the other hand, degree of a subvariety $X \subset \mathbb{C}P^n$ is its intersection with the top power of the fundamental class $[H] = c_1(\mathcal{O}(1))$ of the hyperplane section H. This gives $\deg X = 2ab = c_1(\mathcal{O}(a,b))^2 = 2ab$.

Step 2: Consider the homotopy exact sequence

$$0 \longrightarrow \pi_2(S) \longrightarrow \pi_2(X) \stackrel{\varphi}{\longrightarrow} \pi_1(S^1) \longrightarrow \pi_1(S) \longrightarrow 0$$

for the circle bundle $\pi : S \longrightarrow X$. Since the map φ represents the first Chern class of $\mathcal{O}(1)|_X$, it is obtained as a quotient of $\mathbb{Z}^2$ by a subgroup generated by (a,b), and this map is surjective because a and b are relatively prime. Then $\pi_1(M) = 0$, and Proposition 1 implies that S is diffeomorphic to $S^2 \times S^3$. ■

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THEOREM: There exists two 2-dimensional projective varieties $X_1, X_2 \subset \mathbb{C}P^n$, both biholomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, **such that the cones $C(X_1) \subset \mathbb{C}^{n+1}$ and $C(X_2) \subset \mathbb{C}^{n+1}$ are bi-Lipschitz equivalent, but X_1, X_2 have different degree.**

Proof. Step 1: Consider the link of the singularity $S_i := C(X_i) \cap S^{2n+1}$, where S^{2n+1} is the unit sphere centered in 0. Clearly, a bi-Lipschitz map from S_1 to S_2 induces a bi-Lipschitz map of their cones. Moreover, any diffeomorphism of the sphere S^{2n+1} to itself mapping S_1 to S_2 induces a bi-Lipschitz map of the ambient vector space $\mathbb{C}^{n+1}$, identified with the Riemannian cone of S^{2n+1}, mapping $C(X_1)$ to $C(X_2)$.

Step 2: Any diffeomorphism of a smooth subvariety $Z \subset S^{2n+1}$ to $Z' \subset S^{2n+1}$ can be extended to a diffeomorphism of S^{2n+1} to itself, if $2\dim Z + 1 < 2n+1$. However, dimension of the ambient space can be increased arbitrarily by adding extra variables. Therefore, to prove Theorem it would suffice to find X_1, X_2 such that the corresponding links S_1, S_2 are diffeomorphic. This follows from Proposition 2. ∎