Instanton bundles on $\mathbb{C}P^3$ and special holonomies

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Instanton bundles

DEFINITION: An **mathematical instanton bundle** on $\mathbb{C}P^n$ is a holomorphic bundle E with $c_1(E) = 0$ which satisfies

1. For $n \geqslant 2$, $H^0(E(-1)) = H^n(E(-n)) = 0$.
2. For $n \geqslant 3$, $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$.
3. For $n \geqslant 4$, $H^p(E(k)) = 0$, $2 \leqslant p \leqslant n-2$ and $\forall k$.

REMARK: Mathematical instanton bundles are stable.

DEFINITION: Let $C \subset \mathbb{C}P^n$ be a projective subspace of codimension 2. **Framing** of a holomorphic bundle E is a trivialization of $E|_C$.

The main result of this talk: The moduli space of framed instantons on $\mathbb{C}P^3$ is smooth. The moduli of rank 2 instantons on $\mathbb{C}P^3$ is smooth.

(A joint work with Marcos Jardim).

Plan of the talk.

1. Hyperkähler manifolds, hyperkähler reduction and quiver varieties.

2. Complexification of a hyperkähler manifold and its twistor space.

3. Trihyperkähler reduction and the space of instantons.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

REMARK: The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is holomorphic and symplectic on (M, I).

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: **A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$** (the group of all endomorphisms preserving I, J, K).

Hyperkähler reduction

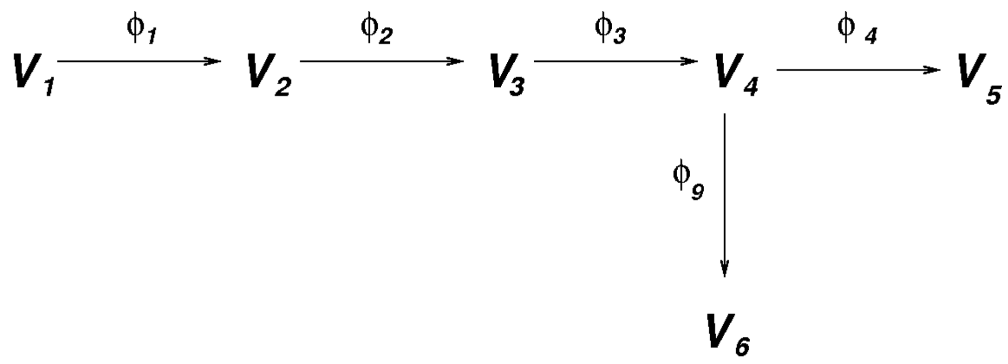
DEFINITION: Let G be a compact Lie group, ρ its action on a hyperkähler manifold M by hyperkähler isometries, and $\mathfrak{g}^*$ a dual space to its Lie algebra. **A hyperkähler moment map** is a G-equivariant smooth map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ such that $\langle \mu_i(v), g\rangle = \omega_i(v, d\rho(g))$, for every $v \in TM$, $g \in \mathfrak{g}$ and $i = 1, 2, 3$, where ω_i is one three Kähler forms associated with the hyperkähler structure.

DEFINITION: Let ξ_1, ξ_2, ξ_3 be three G-invariant vectors in $\mathfrak{g}^*$. The quotient manifold $M /\!\!/\!\!/ G := \mu^{-1}(\xi_1, \xi_2, \xi_3)/G$ is called **the hyperkähler quotient** of M.

THEOREM: (Hitchin, Karlhede, Lindström, Roček)
The quotient $M /\!\!/\!\!/ G$ is hyperkaehler.

Quiver representations

DEFINITION: A **quiver** is an oriented graph. A **quiver representation** is a diagram of complex Hermitian vector spaces and arrows associated with a quiver:

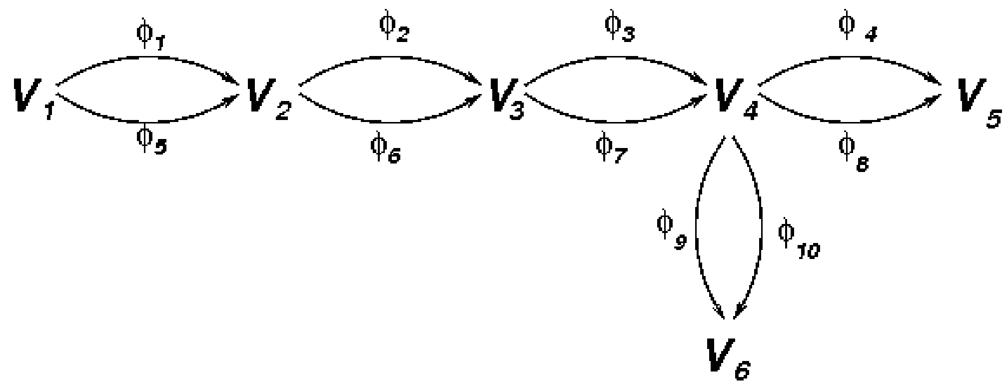


Here, V_i are vector spaces, and φ_i linear maps.

REMARK: If one fixes the spaces V_i, the space of quiver representations is a Hermitian vector space.

Quiver varieties

Starting from a single graph, one can double it up, as follows, obtaining a Nakajima double quiver.



A Nakajima quiver for the Dynkin diagram D_5.

CLAIM: The space M of representations of a Nakajima's double quiver is a quaternionic vector space, and the group $G := U(V_1) \times U(V_2) \times ... \times U(V_n)$ acts on M preserving the quaternionic structure.

DEFINITION: A **Nakajima quiver variety** is a quotient $M /\!\!/\!\!/ G$.

Hyperkähler manifolds as quiver varieties

Many non-compact manifolds are obtained as quiver varieties.

EXAMPLE: A 4-dimensional ALE (asymptotically locally Euclidean) space obtained as a resolution of a du Val singularity, that is, a quotient $\mathbb{C}^2/G$, where $G \subset SU(2)$ is a finite group.

REMARK: Since finite subgroups of $SU(2)$ are classified by the Dynkin diagrams of type A,D,E, these ALE quotients are called **ALE spaces of A-D-E type**.

EXAMPLE: A Hilbert scheme of points on an ALE space of A-D-E type.

EXAMPLE: The moduli asymptotically flat Hermitian Yang-Mills connections on ALE spaces of A-D-E type.

The most important for us example is:

EXAMPLE: **The moduli of framed instantons on $\mathbb{C}P^2$ is a quiver variety,** hence **hyperkähler.** In particular, it is **smooth and connected.**

Complexification of a manifold

DEFINITION: Let M be a complex manifold, equipped with an anticomplex involution ι. The fixed point set $M_{\mathbb{R}}$ of ι is called **a real analytic manifold**, and a germ of M in $M_{\mathbb{R}}$ is called **a complexification** of $M_{\mathbb{R}}$.

QUESTION: What is a complexification of a Kähler manifold (considered as real analytic variety)?

THEOREM: (D. Kaledin, B. Feix) Let M be a real analytic Kähler manifold, and $M_{\mathbb{C}}$ its complexification. **Then $M_{\mathbb{C}}$ admits a hyperkähler structure,** determined uniquely and functorially by the Kähler structure on M.

QUESTION: What is a complexification of a hyperkähler manifold?

ANSWER: Trisymplectic manifolds!

Twistor space

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2+b^2+c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

REMARK: For M compact, $\operatorname{Tw}(M)$ never admits a Kähler structure.

Rational curves on Tw(M).

REMARK: The twistor space **has many rational curves**.

DEFINITION: Denote by Sec(M) **the space of holomorphic sections** of the twistor fibration Tw$(M) \xrightarrow{\pi} \mathbb{C}P^1$.

DEFINITION: For each point $m \in M$, one has **a horizontal section** $C_m := \{m\} \times \mathbb{C}P^1$ of π. The space of horizontal sections is denoted Sec$_{hor}(M) \subset$ Sec(M)

REMARK: The space of horizontal sections of π is identified with M. The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, **some neighbourhood of Sec$_{hor}(M) \subset$ Sec(M) is a smooth manifold of dimension $2 \dim M$.**

DEFINITION: A twistor section $C \subset$ Tw(M) is called **regular**, if $NC = \mathcal{O}(1)^{\dim M}$.

CLAIM: For any $I \neq J \in \mathbb{C}P^n$, consider the evaluation map Sec$(M) \xrightarrow{E_{I,J}} (M, I) \times (M, J)$, $s \longrightarrow s(I) \times s(J)$. Then **$E_{I,J}$ is an isomorphism around the set Sec$_0(M)$ of regular sections.**

Complexification of a hyperkähler manifold.

REMARK: Consider an anticomplex involution $\operatorname{Tw}(M) \xrightarrow{\iota} \operatorname{Tw}(M)$ mapping (m,t) to $(m,i(t))$, where $i: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is a central symmetry. Then $\operatorname{Sec}_{hor}(M) = M$ **is a component of the fixed set of** ι**.**

COROLLARY: $\operatorname{Sec}(M)$ **is a complexification of** M**.**

QUESTION: What are geometric structures on $\operatorname{Sec}(M)$?

Answer 1: For compact M, $\operatorname{Sec}(M)$ **is holomorphically convex** (Stein if $\dim M = 2$).

Answer 2: The space $\operatorname{Sec}_0(M)$ **admits a holomorphic, torsion-free connection** with holonomy $Sp(n,\mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.

Mathematical instantons

DEFINITION: A mathematical instanton on $\mathbb{C}P^3$ is a stable bundle B with $c_1(B) = 0$ and $H^1(B(-1)) = 0$. **A framed instanton** is a mathematical instanton equipped with a trivialization of $B|_\ell$ for some fixed line $\ell = \mathbb{C}P^1 \subset \mathbb{C}P^3$.

DEFINITION: An instanton on $\mathbb{C}P^2$ is a stable bundle B with $c_1(B) = 0$. **A framed instanton** is an instanton equipped with a trivialization $B|_x$ for some fixed point $x \in \mathbb{C}P^2$.

THEOREM: (Atiyah-Drinfeld-Hitchin-Manin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **smooth, connected, hyperkähler.**

THEOREM: (Jardim–V.) The space $\mathbb{M}_{r,c}$ of framed mathematical instantons on $\mathbb{C}P^3$ **is naturally identified with the space of twistor sections** Sec($\mathcal{M}_{r,c}$)**.**

REMARK: This correspondence is not surprising if one realizes that Tw($\mathbb{H}$) $= \mathbb{C}P^3 \backslash \mathbb{C}P^1$.

The space of instantons on $\mathbb{C}P^3$

THEOREM: (Jardim–V.) **The space $\mathbb{M}_{r,c}$ is smooth.**

REMARK: To prove that $\mathcal{M}_{r,c}$ is smooth, one could use hyperkähler reduction. To prove that $\mathbb{M}_{r,c}$ is smooth, we develop **trihyperkähler reduction**, which is **a reduction defined on trisymplectic manifolds.**

We prove that $\mathbb{M}_{r,c}$ **is a trihyperkähler quotient** of a vector space by a reductive group action, hence smooth.

Trisymplectic manifolds

DEFINITION: Let $\mathbf{\Omega}$ be a 3-dimensional space of holomorphic symplectic 2-forms on a complex manifold. Suppose that

- **Ω contains a non-degenerate 2-form**
- For each non-zero degenerate $\Omega \in \mathbf{\Omega}$, one has $\operatorname{rk}\Omega = \frac{1}{2}\dim V$.

Then $\mathbf{\Omega}$ is called **a trisymplectic structure on M**.

REMARK: **The bundles $\ker \Omega$ are involutive,** because Ω is closed.

THEOREM: (Jardim–V.) For any trisymplectic structure on M, M is equipped with a unique holomorphic, torsion-free connection, preserving the forms Ω_i. It is called **the Chern connection** of M.

REMARK: **The Chern connection has holonomy in $Sp(n,\mathbb{C})$ acting on $\mathbb{C}^{2n}\otimes\mathbb{C}^2$.**

Trisymplectic structure on $\operatorname{Sec}_0(M)$

EXAMPLE: Consider a hyperkähler manifold M. Let $I \in \mathbb{C}P^1$, and $ev_I : \operatorname{Sec}_0(M) \longrightarrow (M, I)$ be an evaluation map putting $S \in \operatorname{Sec}_0(M)$ to $S(I)$. Denote by Ω_I the holomorphic symplectic form on (M, I). **Then $ev_I^* \Omega_I$, $I \in \mathbb{C}P^1$ generate a trisymplectic structure.**

COROLLARY: **$\operatorname{Sec}_0(M)$ is equipped with a holomorphic, torsion-free connection with holonomy in $Sp(n, \mathbb{C})$.**

Trihyperkähler reduction

DEFINITION: A trisymplectic moment map $\boldsymbol{\mu}_{\mathbb{C}} : M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \boldsymbol{\Omega}^*$ takes vectors $\Omega \in \boldsymbol{\Omega}, g \in \mathfrak{g} = \operatorname{Lie}(G)$ and maps them to a holomorphic function $f \in \mathcal{O}_M$, such that $df = \Omega \lrcorner g$, where $\Omega \lrcorner g$ denotes the contraction of Ω and the vector field g

DEFINITION: Let $(M, \boldsymbol{\Omega}, S_t)$ be a trisymplectic structure on a complex manifold M. Assume that M is equipped with an action of a compact Lie group G preserving $\boldsymbol{\Omega}$, and an equivariant trisymplectic moment map

$$\boldsymbol{\mu}_{\mathbb{C}} : M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \boldsymbol{\Omega}^*.$$

Let $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$ be the corresponding **level set** of the moment map. Consider the action of the complex Lie group $G_{\mathbb{C}}$ on $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)$. Assume that it is proper and free. Then the quotient $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)/G_{\mathbb{C}}$ is a smooth manifold called **the trisymplectic quotient** of $(M, \boldsymbol{\Omega}, S_t)$, denoted by $M /\!\!/\!\!/\!\!/ G$.

THEOREM: Suppose that the restriction of $\boldsymbol{\Omega}$ to $\mathfrak{g} \subset TM$ is non-degenerate. **Then $M /\!\!/\!\!/\!\!/ G$ trisymplectic.**

Mathematical instantons and the twistor correspondence

REMARK: Using the monad description of mathematical instantons, **we prove that that the map $\operatorname{Sec}_0(\mathcal{M}_{r,c}) \longrightarrow \mathbb{M}_{r,c}$ to the space of mathematical instantons is an isomorphism** (Frenkel-Jardim, Jardim-V.).

REMARK: The smoothness of the space $\operatorname{Sec}_0(\mathcal{M}_{r,c}) = \mathbb{M}_{r,c}$ **follows from the trihyperkähler reduction procedure:**

THEOREM: Let M be flat hyperkähler manifold, and G a compact Lie group acting on M by hyperkähler automorphisms. Suppose that the hyperkähler moment map exists, and the hyperkähler quotient $M/\!\!/\!\!/G$ is smooth. **Then there exists an open embedding**

$$\operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G \xrightarrow{\Psi} \operatorname{Sec}_0(M/\!\!/\!\!/G),$$

which is compatible with the trisymplectic structures on $\operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G$ and $\operatorname{Sec}_0(M/\!\!/\!\!/G)$.

THEOREM: If M is the space of quiver representations which gives $M/\!\!/\!\!/G = \mathcal{M}_{2,c}$, **$\Psi$ gives an isomorphism $\operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G = \operatorname{Sec}_0(M/\!\!/\!\!/G)$.**