Brody lemma and Ahlfors currents

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Plan

1. Kobayashi metrics, Schwartz lemma

2. Normal families, Montel theorem

3. Brody lemma

4. Ahlfors current

Space forms

DEFINITION: Simply connected space form is a homogeneous manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique G-invariant symmetric 2-form:** the standard Euclidean metric.

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

REMARK: From now on, the space forms are assumed to be homogeneous Riemannian manifolds.

Schwartz lemma

CLAIM: (maximum principle) Let f be a holomorphic function defined on an open set U. **Then f cannot have strict maxima in U. If f has non-strict maxima, it is constant.**

EXERCISE: Prove the maximum principle.

LEMMA: (Schwartz lemma) Let $f: \Delta \longrightarrow \Delta$ be a map from disk to itself fixing 0. **Then $|f'(0)| \leqslant 1$, and equality can be realized only if $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.**

Proof: Consider the function $\varphi := \frac{f(z)}{z}$. Since $f(0) = 0$, it is holomorphic, and since $f(\Delta) \subset \Delta$, on the boundary $\partial\Delta$ we have $|\varphi||_{\partial\Delta} \leqslant 1$. Now, **the maximum principle implies that $|f'(0)| = |\varphi(0)| \leqslant 1$,** and equality is realized only if $\varphi = const$. ■

Conformal automorphisms of the disk

CLAIM: Let $\Delta \subset \mathbb{C}$ be the unit disk. **Then the group $\operatorname{Aut}(\Delta)$ of its holomorphic automorphisms acts on Δ transitively.**

Proof: Let $V_a(z) = \frac{z-a}{1-\overline{a}z}$ for some $a \in \Delta$. Then $V_a(0) = -a$. To prove transitivity, it remains to show that $V_a(\Delta) = \Delta$, which is implied from

$$|V_a(z)| = |V_a(z)||z| = \left|\frac{z\overline{z} - a\overline{z}}{1 - \overline{a}z}\right| = \left|\frac{1 - a\overline{z}}{1 - \overline{a}z}\right| = 1.$$

■

REMARK: The group $PU(1,1) \subset PGL(2,\mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form h of signature (1,1) acts on a disk $\{l \in \mathbb{C}P^1 \mid h(l,l) > 0\}$ by holomorphic automorphisms.

COROLLARY: Let $\Delta \subset \mathbb{C}$ be the unit disk, $\operatorname{Aut}(\Delta)$ the group of its conformal automorphisms, and $\Psi: PU(1,1) \longrightarrow \operatorname{Aut}(\Delta)$ the map constructed above. **Then Ψ is an isomorphism.**

COROLLARY: Let h be a homogeneous metric on $\Delta = PU(1,1)/S^1$. **Then (Δ, h) is conformally equivalent to $(\Delta,$ flat metric$)$.**

Upper half-plane

REMARK: The map $z \longrightarrow -\sqrt{-1}\,(z-1)^{-1}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}$.

PROPOSITION: The group $\operatorname{Aut}(\Delta)$ acts on the upper half-plane $\mathbb{H}$ as $z \xrightarrow{A} \frac{az+b}{cz+d}$, where $a,b,c,d \in \mathbb{R}$, and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$.

REMARK: The group of such A is naturally identified with $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$. Since $PSL(2,\mathbb{R})$ acts on its Lie algebra preserving the Killing form, $PSL(2,\mathbb{R})$ embeds to $SO(1,2)$. Both of these groups are 3-dimensional, since they are isomorphic.

REMARK: We have shown that $\mathbb{H} = SO(1,2)/S^1$. This gives a **natural isomorphism of $\mathbb{H}$ and the hyperbolic space.** Under this isomorphism, **holomorphic automorphisms correspond to isometries.**

Poincaré metric on disk

DEFINITION: Poincaré metric on a unit disk $\Delta \subset \mathbb{C}$ is an Aut(Δ)-invariant metric **(it is unique up to a constant multiplier)**.

DEFINITION: Let $f: M \longrightarrow M_1$ be a map of metric spaces. Then f is called **C-Lipschitz** if $d(x,y) \geqslant Cd(f(x), f(y))$. A map is called **Lipschitz** if it is C-Lipschitz for some $C > 0$.

THEOREM: (Schwartz-Pick lemma)
Any holomorphic map $\varphi: \Delta \longrightarrow \Delta$ from a unit disk to itself is 1-Lipschitz with respect to Poicaré metric.

Proof. Step 1: We need to prove that for each $x \in \Delta$ the norm of the differential satisfies $|D\varphi_x| \leqslant 1$. Since the automorphism group acts on Δ transitively, **it suffices to prove that $|D\varphi_x| \leqslant 1$ when $x = 0$ and $\varphi(x) = 0$.**

Step 2: This is Schwartz lemma. ■

Kobayashi pseudometric

DEFINITION: Pseudometric on M is a function $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric: $d(x,y) = d(y,x)$ and satisfies the triangle inequality $d(x,y) + d(y,z) \geqslant d(x,z)$.

REMARK: Let $\mathfrak{D}$ be a set of pseudometrics. **Then $d_{\max}(x,y) := \sup_{d\in\mathfrak{D}} d(x,y)$ is also a pseudometric.**

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M is $d_{\max}$ for the set $\mathfrak{D}$ of all pseudometrics such that any holomorphic map from the Poincaré disk to M is distance-decreasing.

EXERCISE: Prove that **the distance between points x, y in Kobayashi pseudometric is infimum of the Poincaré distance over all sets of Poincaré disks connecting x to y.**

EXAMPLE: The Kobayashi pseudometric on $\mathbb{C}$ vanishes.

CLAIM: Any holomorphic map $X \xrightarrow{\varphi} Y$ is 1-Lipschitz with respect to the Kobayashi pseudometric.

Proof: If $x \in X$ is connected to x' by a sequence of Poincare disks $\Delta_1, ..., \Delta_n$, then $\varphi(x)$ is connected to $\varphi(x')$ by $\varphi(\Delta_1), ..., \varphi(\Delta_n)$. ■

Kobayashi hyperbolic manifolds

COROLLARY: Let $B \subset \mathbb{C}^n$ be a unit ball, and $x, y \in B$ points with coordinates $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Since x_i, y_i belongs to Δ, it makes sense to compute the Poincare distance $d_P(x_i, y_i)$. **Then $d_K(x, y) \geqslant \max_i d_P(x_i, y_i)$.**

Proof: Each of projection maps $\Pi_i : B \longrightarrow \Delta$ is 1-Lipschitz. ∎

DEFINITION: A variety is called **Kobayashi hyperbolic** if the Kobayashi pseudometric d_K is non-degenerate.

DEFINITION: A **domain** in $\mathbb{C}^n$ is an open subset. A **bounded domain** is an open subset contained in a ball.

COROLLARY: Any bounded domain Ω in $\mathbb{C}^n$ is Kobayashi hyperbolic.

Proof: Without restricting generality, we may assume that $\Omega \subset B$ where B is an open ball. Then the Kobayashi distance in Ω is $\geqslant$ that in B. However, the Kobayashi distance in B is bounded by the metric $d(x, y) := \max_i d_P(x_i, y_i)$ as follows from above. ∎

Caratheodory metric

DEFINITION: Let $x, y \in M$ be points on a complex manifold. Define **Caratheodory pseudometric** as $d_C(x,y) = \sup\{d_P(f(x), f(y))\}$, where the supremum is taken over all holomorphic map $f: M \longrightarrow \Delta$, and d_P is Poincare metric on the disk Δ.

REMARK: Usually the term "Kobayashi/Caratheodory pseudometric" is abbreviated to "Kobayashi/Caratheodory metric", **even when it is not a metric.**

REMARK: Caratheodory pseudometric **satisfies the triangle inequality** because a supremum of pseudometrics satisfies triangle inequality.

Exercise: Prove that **Caratheodory pseudometric is bounded by the Kobayashi pseudometric:** $d_K \geqslant d_C$.

REMARK: Clearly, $d_C \neq 0$ on any bounded domain.

Complex hyperbolic space

DEFINITION: Let $V = \mathbb{C}^{n+1}$ be a complex vector space equipped with a Hermitian metric h of signature $(1,n)$, and $\mathbb{H}^n_{\mathbb{C}} \subset \mathbb{P}V$ projectivization of the set of positive vectors $\{x \in V \quad h(x,\bar{x}) > 0\}$. Then $\mathbb{H}^n_{\mathbb{C}}$ is equipped with a homogeneous action of $U(1,n)$. The same argument as used for space forms implies that $\mathbb{H}^n_{\mathbb{C}}$ admits a $U(1,n)$-invariant Hermitian metric, which is unique up to a constant multiplier. This Hermitian complex manifold is called **complex hyperbolic space**.

REMARK: For $n > 1$ **it is not isometric, to the real hyperbolic spaces defined earlier.**

REMARK: As a complex manifold $\mathbb{H}^n_{\mathbb{C}}$ **is isomorphic to an open ball in $\mathbb{C}^n$.**

REMARK: The Kobayashi metric and the Caratheodory metric on $\mathbb{H}^n_{\mathbb{C}}$ are $U(1,n)$-invariant, because $U(1,n)$ acts holomorphically, hence proportional to the hyperbolic metric, which is also called **Bergman metric** on an open ball.

Exercise: Prove that **Kobayashi metric on a ball in $\mathbb{C}^n$ is equal to the Caratheodory metric.**

Uniform convergence for Lipschitz maps

DEFINITION: A sequence of maps $f_i: M \longrightarrow N$ between metric spaces **uniformly converges** (or **converges uniformly on compacts**) to $f: M \longrightarrow N$ if for any compact $K \subset M$, we have $\lim\limits_{i\to\infty} \sup_{x\in K} d(f_i(x), f(x)) = 0$.

Claim 1: Suppose that a sequence $f_i: M \longrightarrow N$ of 1-Lipschitz maps converges to f pointwise in a countable dense subset $M' \subset M$. **Then f_i converges to f uniformly on compacts.**

Proof: Let $K \subset M$ be a compact set, and $N_\varepsilon \subset M'$ a finite subset such that K is a union of ε-balls centered in N_ε (such N_ε is called **an ε-net**). Then there exists N such that $\sup_{x\in N_\varepsilon} d(f_{N+i}(x), f(x)) < \varepsilon$ for all $i \geqslant 0$. Since f_i are 1-Lipschitz, this implies that

$$\sup_{y\in K} d(f_{N+i}(y), f(y)) \leqslant$$
$$\leqslant \sup_{x\in N_\varepsilon} d(f_{N+i}(x), f(x)) + \inf_{x\in N_\varepsilon} (d(f_{N+i}(x), y) + d(f(x), y)) \leqslant 3\varepsilon.$$

■

Exercise: Prove that the limit f is also 1-Lipschitz.

REMARK: This proof works when M is a pseudo-metric space, as long as N is a metric space.

Arzelà-Ascoli theorem for Lipschitz maps

DEFINITION: Let M, N be metric spaces. A subset $B \subset M$ is **bounded** if it is contained in a ball. A family $\{f_\alpha\}$ of functions $f_\alpha : M \longrightarrow N$ is called **uniformly bounded on compacts** if for any compact subset $K \subset M$, there is a bounded subset $C_K \subset N$ such that $f_\alpha(K) \subset C_K$ for any element f_α of the family.

THEOREM: (Arzelà-Ascoli for Lipschitz maps) Let $\mathcal{F} := \{f_\alpha\}$ be an infinite uniformly bounded set of 1-Lipschitz maps $f_\alpha : M \longrightarrow \mathbb{C}$, where M is a pseudo-metric space. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly.**

REMARK: The limit f is clearly also 1-Lipschitz.

Proof. Step 1: Suppose we can prove Arzelà-Ascoli when M is compact. Then we can choose a sequence of compact subsets $K_i \subset M$, find subsequences in $\mathcal{F}$ converging on each K_i, and use the diagonal method to find a subsequence converging on all K_i. Therefore, **we can assume that M is bounded, and all maps $f_\alpha : M \longrightarrow \mathbb{C}$ map M into a compact subset $N \subset \mathbb{C}$.**

Arzelà-Ascoli theorem for Lipschitz maps (2)

THEOREM: (Arzelà-Ascoli for Lipschitz maps)
Let $\mathcal{F} := \{f_\alpha\}$ be an infinite uniformly bounded set of 1-Lipschitz maps $f_\alpha : M \longrightarrow \mathbb{C}$, where M is a pseudo-metric space. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly.**

REMARK: The limit f is clearly also 1-Lipschitz.

Proof. Step 1: We can assume that M is compact, and all maps $f_\alpha : M \longrightarrow \mathbb{C}$ map M into a compact subset $N \subset \mathbb{C}$.

Step 2: Find a dense, countable subset $Z \subset M$. Using diagonal method, **find a sequence $\{f_i\} \subset \mathcal{F}$ converging pointwise to some f at all $z \in Z$.**

Step 3: Being a pointwise limit of Lipschitz functions, $f|_Z$ **is also Lipschitz, and f_i converge to f uniformly on Z.**

Step 4: Since a Lipschitz function maps Cauchy sequences to Cauchy sequences, it can be extended to a Lipschitz function on the completion $\overline{M}$.
■

Normal families of holomorphic functions

DEFINITION: Let M be a complex manifold. A family $\mathcal{F} := \{f_\alpha\}$ of holomorphic functions $f_\alpha : \ M \longrightarrow \mathbb{C}$ is called **normal family** if $\mathcal{F}$ is uniformly bounded on compact subsets.

THEOREM: (Montel's theorem)
Let M be a complex manifold with countable base, and $\mathcal{F}$ a normal, infinite family of holomorphic functions. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : \ M \longrightarrow \mathbb{C}$ uniformly,** and f is holomorphic.

Proof. Step 1: As in the first step of Arzelà-Ascoli, it suffices to prove Montel's theorem on a subset of M where $\mathcal{F}$ is bounded. Therefore, **we may assume that all f_α map M into a disk Δ.**

Step 2: All f_α are 1-Lipschitz with respect to Kobayashi metric. Therefore, **Arzelà-Ascoli theorem can be applied, giving a uniform limit $f = \lim f_i$.**

Step 3: A uniform limit of holomorphic functions is holomorphic by Cauchy formula. ■

REMARK: The sequence $f = \lim f_i$ converges uniformly with all derivatives, again by Cauchy formula.

Normal families in complete generality

DEFINITION: A set of holomorphic maps $f_\alpha : X \longrightarrow Y$ is called **a normal family** if any sequence $\{f_i\}$ in $\{f_\alpha\}$ has a subsequence converging unformly on compacts.

THEOREM: Let $f_\alpha : X \longrightarrow Y$ be a family of holomorphic maps such that for any point $x \in X$ there exists its neighbourhood with compact closure $K \subset X$ and a Kobayashi hyperbolic open subset $V_K \subset Y$ such that all f_α map K to V_k. **Then f_α is a normal family.**

Proof: $f_\alpha|_K$ is Lipschitz with respect to the Kobayashi metric, and Arzelà-Ascoli theorem can be applied. ■

Brody curves and Brody maps

DEFINITION: Let M be a complex Hermitian manifold. **Brody curve** is a non-constant holomorphic map $f: \mathbb{C} \longrightarrow M$ such that $|df| \leqslant C$ for some constant C. Here $|df|$ is understood as an operator norm of $df: T_z\mathbb{C} \longrightarrow TM$, where $\mathbb{C}$ is equipped with the standard Euclidean metric.

DEFINITION: Let (Δ_r, g_r) be a disk of radius r in $\mathbb{C}$ with the Poincare metric g_r, rescaled in such a way that the unit tangent vector to 0 has length 1. **Brody map** to a Hermitian complex manifold is a map $f: \Delta_r \longrightarrow M$ such that $|df| \leqslant 1$ (here the operator norm is taken with respect to the Poincare metric on Δ_r) and $|df|(z) = 1$ at $z = 0$.

Lemma 1: Let $f_r: \Delta_r \longrightarrow M$ be a sequence of Brody maps with $r \longrightarrow \infty$. **Then f_r converges uniformly to a Brody curve f satisfying $|df|(z) = 1$ at $z = 0$.**

Proof. Step 1: Let $r_1 < r_2$. **The identity map $\tau: (\Delta_{r_1}, g_1) \longrightarrow (\Delta_{r_2}, g_2)$ is 1-Lipschitz.** Indeed, it is 1-Lipschitz with respect to the usual Poincare metric: $\tau^*(r_2^{-2} g_2) \leqslant r_1^{-2} g_1$. Since $r_1 < r_2$, this gives $\tau^* g_2 \leqslant g_1$.

Brody curves and Brody maps (2)

Lemma 1: Let $f_r : \Delta_r \longrightarrow M$ be a sequence of Brody maps with $r \longrightarrow \infty$. **Then f_r converges uniformly to a Brody curve f satisfying $|df|(z) = 1$ at $z = 0$.**

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Step 2: Restricted on any disk Δ_R, the family $\{f_r, r > R\}$ is a normal family (it is Lipschitz), hence converges uniformly to a Lipschitz map. Since a uniform limit of holomorphic maps is holomorphic, **the family $\{f_r\big|_{\Delta_R}, r > R\}$ converges to a holomorhic map on $\varphi_R : \Delta_R \longrightarrow M$.**

Step 3: The map φ_R is Lipschitz with respect to all metrics g_r, $r > R$. Since $\lim_r g_r$ is the standard Euclidean metric g_∞, **φ_R is Lipschitz with respect to g_∞.**

Step 4: $\lim_R \varphi_R$ converges to a holomorphic Lipschitz map $\mathbb{C} \longrightarrow M$. Since all f_r and φ_R satisfy $|d\varphi_R|(z) = 1$ at $z = 0$, the same is true for the limit. ∎

Brody curves and Brody maps

THEOREM: (Brody lemma)
Let M be a compact complex manifold which is not Kobayashi hyperbolic. **Then M contains a Brody curve.**

Let us equip M with a Hermitian metric h. If $|df|(0) \leqslant C$ for any holomorphic map $(\Delta_1, g_1) \longrightarrow M$, then the Kobayashi metric satisfies $d_K \geqslant C^{-1}h$, and M is Kobayashi hyperbolic. If it is non-bounded, we can always rescale the disc to obtain a map $f_r: (\Delta_r, g_r) \longrightarrow M$ with $r = |df|(0)$, and then $|df_r|(0) = 1$.

Then Brody lemma follows from Lemma 1 and the following lemma.

LEMMA: Let M be a compact Hermitian manifold, and $\psi_r: (\Delta_r, g_r) \longrightarrow M$ a sequence of holomorphic maps satisfying $|d\psi_r|(0) \geqslant 1$, $r \longrightarrow \infty$. **Then there exists a sequence of Brody maps $f_s: (\Delta_s, g_s) \longrightarrow M$, with $s \longrightarrow \infty$.**

Brody curves and Brody maps (2)

LEMMA: Let M be a compact Hermitian manifold, and $\psi_r : (\Delta_r, g_r) \longrightarrow M$ a sequence of holomorphic maps satisfying $|d\psi_r|(0) \geqslant 1$, $r \longrightarrow \infty$. **Then there exists a sequence of Brody maps $f_s : (\Delta_s, g_s) \longrightarrow M$, with $s \longrightarrow \infty$.**

Proof. Step 1: We need to construct a sequence of Brody maps, which are 1-Lipschitz maps $f_s : (\Delta_s, g_s) \longrightarrow M$, with $|df_s|(0) = 1$. The identity map

$$\Psi_{r-\varepsilon,r} : (\Delta_{r-\varepsilon}, g_{r-\varepsilon}) \longrightarrow (\Delta_r, g_r)$$

is 1-Lipschitz, and satisfies

$$\lim_{z \longrightarrow \partial\Delta_{r-\varepsilon}} |d\Psi_{r-\varepsilon,r}|(z) = 0.$$

Let $u := r-\varepsilon$ and $\tilde{f}_u := \Psi_{r-\varepsilon,r} \circ \psi_r$ be a restriction of f_r to the disk $(\Delta_{r-\varepsilon}, g_{r-\varepsilon})$. Then f_u is also Lipschitz and $|d\tilde{f}_u|$ reaches maximum at a point z_u somewhere inside the disk Δ_u.

Step 2: Applying appropriate holomorphic isometry of Δ_u, we may assume that $|d\tilde{f}_u|(z)$ takes maximum $C_u \geqslant 1$ for $z = 0$. Rescaling $\tilde{f}_u$, and putting $s := C_u u$, we obtain a map $f_s : \Delta_s \longrightarrow M$ which is 1-Lipschitz and satisfies $|df_s| \leqslant 1$, $|df_s|(0) = 1$. ■

Currents and generalized functions

DEFINITION: Let F be a Hermitian bundle with connection ∇, on a Riemannian manifold M with Levi-Civita connection, and

$$\|f\|_{C^k} := \sup_{x\in M}\left(|f| + |\nabla f| + ... + |\nabla^k f|\right)$$

the corresponding **C^k-norm** defined on smooth sections with compact support. **The C^k-topology is independent from the choice of connection and metrics.**

DEFINITION: A generalized function is a functional on top forms with compact support, which is continuous in one of C^i-topologies.

DEFINITION: A k-current is a functional on $(\dim M - k)$-forms with compact support, which is continuous in one of C^i-topologies.

REMARK: Currents are forms with coefficients in generalized functions.

Currents on complex manifolds

DEFINITION: The space of currents is equipped with **weak topology** (a sequence of currents converges if it converges on all forms with compact support).

CLAIM: De Rham differential is continuous on currents, and the Poincare lemma holds. Hence, **the cohomology of currents are the same as cohomology of smooth forms.**

DEFINITION: On an complex manifold, **(p,q)-currents** are (p,q)-forms with coefficients in generalized functions

REMARK: In the literature, this is sometimes called $(n-p,n-q)$-currents.

CLAIM: The Poincare and Poincare Dolbeault-Grothendieck lemma hold on (p,q)-currents, and **the d- and $\overline{\partial}$-cohomology are the same as for forms.**

Positive currents

REMARK: Positive generalized functions are all C^0-continuous as functionals on $C^\infty M$. A positive generalized function multiplied by a positive volume form **gives a measure on a manifold,** and all measures are obtained this way.

DEFINITION: Let $\dim_{\mathbb{C}} M = n$. **The cone of positive $(n-1, n-1)$-currents** is generated by $\alpha(-\sqrt{-1}\,)^{n-1}\alpha \prod_{i=1}^{n-1} dz_i \wedge d\bar{z}_i$, where α is a non-negative generalized function (that is, a measure), and z_i holomorphic functions.

REMARK: An $(n-1, n-1)$-current α on an n-dimensional complex manifold is positive if and only if $\int_M \alpha \wedge \beta \geqslant 0$, where $\beta = (-\sqrt{-1}\,)^1 \alpha dz \wedge d\bar{z}$, z a holomorphic function, and α a smooth non-negative function with compact support.

EXAMPLE: A current of integration $\beta \longrightarrow \int_Z \beta$ is positive, for any 1-dimensional complex subvariety $Z \subset M$.

REMARK: If Z is without boundary, the current of integration C_Z is closed by Stokes' theorem. **If Z has boundary, we have**

$$\langle dC_z, \beta\rangle = \int_{\mathbb{Z}} d\beta = \int_{\partial Z} \beta,$$

and this is usually non-zero.

Ahlfors currents

THEOREM: Let $\varphi: \mathbb{C} \longrightarrow M$ be a Brody curve on a complex Hermitian manifold, and $\Delta_r \subset M$ the corresponding disk embeddings. Denote by $A(r)$ the area of Δ_r in M, and let C_{Δ_r} be its current of integration. **Then there exists a sequence r_i such that $\lim_i A(r^{-1}) C_{\Delta_r}$ converges to a closed current.**

REMARK: Any of such limits is called **Ahlfors current**. It is positive, closed, non-zero $(n-1, n-1)$ current, which can be understood as "the current of integration" along the Brody curve.

Proof. Step 1: Let $l(r)$ be the length of $\partial\Delta_r$. Using

$$\langle dC_z, \beta \rangle = \int_{\mathbb{Z}} d\beta = \int_{\partial Z} \beta,$$

we obtain that **it suffices to show that $\lim_i \frac{l(r_i)}{A(r_i)} = 0$ for an appropriate sequence r_i.**

Ahlfors currents (2)

THEOREM: Let $\varphi: \mathbb{C} \longrightarrow M$ be an entire curve on a complex Hermitian manifold, and $\Delta_r \subset M$ the corresponding disk embeddings. Denote by $A(r)$ the area of Δ_r in M, and let C_{Δ_r} be its current of integration. **Then there exists a sequence r_i such that $\lim_i A(r^{-1}) C_{\Delta_r}$ converges to a closed current.**

Step 1: Let $l(r)$ be the length of $\partial\Delta_r$. Then **it suffices to show that $\lim_i \frac{l(r_i)}{A(r_i)} = 0$ for an appropriate sequence r_i.**

Step 2: Consider the function $f(x) = |d\varphi|(x)$ on $\mathbb{C}$. Then $A(r) = \int_{\Delta_r} f^2$ and $l(r) = \int_{\partial\Delta_r} f$ (from now on, all integrals are taken with respect to the usual area and length Lebesgue measure on $\mathbb{C}$ and $\partial\Delta_r$). **If such $\{r_i\}$ does not exist, we obtain that $l(r)/A(r) > C$ for some constant $C > 0$.**

Step 3: Since φ is conformal, the volume of a thin strip $\Delta_r \backslash \Delta_{r-\varepsilon} \subset M$ is approximately equal to $\varepsilon \int_{\partial\Delta_r} f^2$. This gives $\int_{\partial D_r} f^2 = A'(r)$.

Step 4: Now we can forget about M entirely. **We are given a positive, bounded function f on $\mathbb{C}$ which satisfies $\int_{\partial D_r} f^2 = A'(r)$, $\int_{\partial D_r} f = l(r)$, and $l(r)/A(r) > C$.** We need to show that this is impossible.

Ahlfors currents (3)

Step 4: Now we can forget about M entirely. **We are given a positive, bounded function f on $\mathbb{C}$ which satisfies $\int_{\partial D_r} f^2 = A'(r)$, $\int_{\partial D_r} f = l(r)$, and $l(r)/A(r) > C$.** We need to show that this is impossible.

Step 5: Using Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$\left(\int_{\partial D_r} f\right)^2 = l(r)^2 \leqslant 2\pi r \int_{\partial D_r} f^2 = 2\pi r A'(r).$$

Then $l(r) \geqslant CA(r)$ gives $C^2 A^2(r) \leqslant 2\pi r A'(r)$. Writing $C_1 = C^2(2\pi)^{-1}$, we obtain $rA'(r) \geqslant A(r)^2 C_1$.

Step 6: We have

$$\left(\frac{1}{-A(r)}\right)' = \frac{A'(r)}{A^2(r)} \geqslant \frac{C_1}{r}$$

Integrating both sides, we get

$$-\frac{1}{A(r)} \geqslant C_1 \log(r) - C_2$$

which is impossible, because $A(r)$ is monotonous. ■