

# **Holomorphic bundles on foliations**

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## Gauduchon metrics

**DEFINITION:** A Hermitian metric  $\omega$  on a complex  $n$ -manifold is called **Gauduchon** if  $dd^c\omega^{n-1} = 0$ .

**THEOREM:** (P. Gauduchon, 1978) Let  $M$  be a compact, complex manifold, and  $h$  a Hermitian form. **Then there exists a Gauduchon metric conformally equivalent** to  $h$ , and it is unique, up to a constant multiplier.

**REMARK:** If  $\omega$  is Gauduchon, then (by Stokes' theorem)  $\int_M \omega^{n-1} dd^c f = 0$  for any  $f$ . The curvature  $\Theta_L$  of a holomorphic line bundle  $L$  is well-defined up to  $dd^c \log |h|$ , where  $h$  is a conformal factor. Therefore, **for any line bundle  $L$ , the quantity  $\deg_\omega L := \int_M \omega^{n-1} \wedge \Theta_L$  is well defined.**

**REMARK:** Unlike the Kähler case,  $\deg_\omega L$  is a holomorphic invariant of  $L$ , and **not topological.**

**DEFINITION:** Given a torsion-free coherent sheaf  $F$  of rank  $r$ , let  $\det F := \Lambda^r F^{**}$ . From algebraic geometry it is known that  $\det F$  is a line bundle. Define **the degree**  $\deg_\omega F := \deg_\omega \det F = \int_M \text{Tr } \Theta_F \wedge \omega^{n-1}$ .

## Kobayashi-Hitchin correspondence

**DEFINITION:** Let  $F$  be a coherent sheaf over an  $n$ -dimensional Gauduchon manifold  $(M, \omega)$ , and  $\text{slope}(F) := \frac{\deg_{\omega} F}{\text{rank}(F)}$ . A torsion-free sheaf  $F$  is called **stable** if for all subsheaves  $F' \subset F$  one has  $\text{slope}(F') < \text{slope}(F)$ . If  $F$  is a direct sum of stable sheaves of the same slope,  $F$  is called **polystable**.

**DEFINITION:** A Hermitian metric on a holomorphic vector bundle  $B$  is called **Yang-Mills** (Hermitian-Einstein) if  $\Theta_B \wedge \omega^{n-1} = \text{slope}(F) \cdot \text{Id}_B \cdot \omega^n$ , where  $\Theta_B$  is its curvature.

**THEOREM:** (**Kobayashi-Hitchin correspondence**; Donaldson, Buchsdaahl, Uhlenbeck-Yau, Li-Yau, Lübke-Teleman): Let  $B$  be a holomorphic vector bundle. **Then  $B$  admits Yang-Mills metric if and only if  $B$  is polystable.**

**COROLLARY:** Any tensor product of polystable bundles is polystable.

**REMARK:** This result **was generalized to coherent sheaves** by Bando and Siu.

**REMARK:** **Stability is required** if you want to classify vector bundles or construct their moduli spaces.

## Positivity for stable bundles

“Bogomolov’s inequality”: if  $\deg B = 0$  and  $B$  is Yang-Mills, then  $\text{Tr}(\Theta_B \wedge \Theta_B) \wedge \omega^{n-2}$  is a positive volume form, vanishing only if the curvature  $\Theta_B$  of  $B$  vanishes.

**COROLLARY:** A stable bundle on a Kähler manifold with  $c_1(B) = 0, c_2(B) = 0$  is flat.

Today I will give a version of this statement on foliated manifolds. This would give, in particular, the notion of “Higgs bundles” on foliated manifolds, and the “Hitchin-Simpson correspondence” between flat bundles and Higgs bundles.

**REMARK:** For elliptic fibrations this is done in *“Stable Higgs bundles over positive principal elliptic fibrations”* (arXiv:1806.03838, Indranil Biswas, Mahan Mj, M. V.)

## Transversally Kähler foliations

**DEFINITION: Semi-Hermitian form** is a form  $\omega \in \Lambda^{1,1}(M, \mathbb{R})$  such that  $\omega(Ix, x) \geq 0$  for any  $x \in TM$  (the inequality is strict iff  $\omega$  is Hermitian).

**DEFINITION:** In this talk, a **foliation** on a complex manifold  $M$  is a complex sub-bundle  $F \subset TM$ ,  $\dim_{\mathbb{R}} F = 2$ , closed under commutator (usually it is assumed to be holomorphic). A foliation is called **transversally Kähler** if  $M$  is equipped with a closed semi-Hermitian form  $\omega_0$  such that  $\omega_0(x, \cdot) = 0$  for any  $x \in F$  and  $\omega_0$  is Hermitian on  $TM/F$ .

**REMARK:** On a compact Kähler  $n$ -manifold  $(M, \omega)$ , a semi-Hermitian form  $\omega_0$  is never exact. Indeed,  $\int_M \omega_0 \wedge \omega^{n-1} > 0$ , hence  $\omega_0$  cannot be exact. On compact, complex, non-Kähler manifolds, transversally Kähler foliations with exact  $\omega_0$  are quite common.

**EXAMPLE: The classical Hopf surface** is  $H := \mathbb{C}^2 \setminus 0 / \mathbb{Z}$ , where  $\mathbb{Z}$  acts as a multiplication by a complex number  $\lambda$ ,  $|\lambda| > 1$ . Clearly,  $H$  is diffeomorphic to  $S^1 \times S^3$ , and fibered over  $\mathbb{C}P^1$  with fiber  $\mathbb{C}^* / \langle \lambda \rangle$ .

**CLAIM:** Let  $\pi : H \rightarrow \mathbb{C}P^1$  be the standard projection, and  $\omega_0 := \pi^* \omega_{\mathbb{C}P^1}$  be a pullback of the Fubini-Study form. Clearly,  $\omega_0$  is exact, because  $H^2(H) = 0$  (by Künneth formula). Therefore,  $H$  admits a transversally Kähler, exact form.

## Locally trivial elliptic fibrations.

**DEFINITION:** A **principal elliptic fibration**  $M$  is a complex manifold equipped with a free holomorphic action of a 1-dimensional compact complex torus  $T$ .

Such a manifold is fibered over  $M/T$ , with fiber  $T$ .

**REMARK:** It is a principal  $T$ -bundle: all fibers are identified with  $T$ , with  $T$  acting on fibers freely.

**DEFINITION:** Let  $M \xrightarrow{\pi} X$  be a principal elliptic fibration,  $M$  compact. We say that  $M$  is **positive elliptic fibration**, if for some Kähler class  $\omega$  on  $X$ ,  $\pi^*\omega$  is exact. (**“Kähler class” is a cohomology class of a Kähler form**).

**EXAMPLE:** The classical Hopf surface introduced earlier.

**EXAMPLE:** A more general example is given by  $\text{Tot}(L^*)/\langle \mathbb{Z} \rangle$ , where  $L$  is an ample line bundle. Such manifold is called **a regular Vaisman manifold**. It is positive, because  $\pi^*(c_1(L)) = 0$ , and  $c_1(L)$  is a Kähler class.

## Calabi-Eckmann manifolds

### Calabi-Eckmann manifolds.

Fix  $\alpha \in \mathbb{C}$ ,  $\alpha$  non-real,  $|\alpha| > 1$ . Consider a subgroup

$$G := \{e^t \times e^{\alpha t} \subset \mathbb{C}^* \times \mathbb{C}^*, \quad t \in \mathbb{C}\} \subset \mathbb{C}^* \times \mathbb{C}^*$$

within  $\mathbb{C}^* \times \mathbb{C}^*$ . It is clearly co-compact and closed, with  $\mathbb{C}^* \times \mathbb{C}^*/G$  being an elliptic curve  $\mathbb{C}^*/\langle\alpha\rangle$ .

Now, let  $M := (\mathbb{C}^n \setminus 0) \otimes (\mathbb{C}^m \setminus 0)/G$ , with  $G \subset \mathbb{C}^* \times \mathbb{C}^*$  acting on  $(\mathbb{C}^n \setminus 0) \otimes (\mathbb{C}^m \setminus 0)$  by  $(t_1, t_2)(x, y) \rightarrow (t_1 x, t_2 y)$ . Clearly,  $M$  is fibered over

$$\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1} = (\mathbb{C}^n \setminus 0) \otimes (\mathbb{C}^m \setminus 0)/\mathbb{C}^* \times \mathbb{C}^*$$

with a fiber  $\mathbb{C}^* \times \mathbb{C}^*/G$ , which is an elliptic curve. Its total space  $M$  is called **the Calabi-Eckmann manifold**. It is diffeomorphic to  $S^{2n-1} \times S^{2m-1}$ .

**REMARK:** The map  $M \rightarrow \mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$  **is a principal elliptic fibration**.

**REMARK:** The pullback of a Kähler form from  $\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$  to  $M$  **is exact**, because  $H^2(M) = 0$  (by Künneth formula).

## Irregular and quasi-regular foliations

**DEFINITION:** A foliation is called **quasi-regular** if all its leaves are compact. If this is not so, it is called **irregular**. A foliation is called **regular** if all its leaves are compact, and the leaf space is smooth.

**REMARK:** The examples given above (Vaisman, Calabi-Eckmann) **are deformed naturally to irregular foliated transversally Kähler manifolds.**

**REMARK:** Calabi-Eckmann manifolds were generalized by Lopez de Medrano, Verjovsky and Meersseman. The complex structure on Calabi-Eckmann can be deformed together with the foliation, giving **a transversally Kähler manifold with a foliation having non-compact leaves** (“LVM-manifolds”).



## Oeljeklaus-Toma manifolds

Let  $K$  be a number field which has  $2t$  complex embeddings denoted  $\tau_i, \bar{\tau}_i$  and  $s$  real ones denoted  $\sigma_i$ ,  $s > 0$ ,  $t > 0$ .

Let  $\mathcal{O}_K^{*,+} := \mathcal{O}_K^* \cap \bigcap_i \sigma_i^{-1}(\mathbb{R}^{>0})$ . Choose in  $\mathcal{O}_K^{*,+}$  a free abelian subgroup  $\mathcal{O}_K^{*,U}$  of rank  $s$  such that the quotient  $\mathbb{R}^s / \mathcal{O}_K^{*,U}$  is compact, where  $\mathcal{O}_K^{*,U}$  is mapped to  $\mathbb{R}^t$  as  $\xi \rightarrow (\log(\sigma_1(\xi)), \dots, \log(\sigma_t(\xi)))$ . Let  $\Gamma := \mathcal{O}_K^+ \rtimes \mathcal{O}_K^{*,U}$ .

**DEFINITION:** An Oeljeklaus-Toma manifold is a quotient  $\mathbb{C}^t \times H^s / \Gamma$ , where  $\mathcal{O}_K^+$  acts on  $\mathbb{C}^t \times H^t$  as

$\zeta(x_1, \dots, x_t, y_1, \dots, y_s) = (x_1 + \tau_1(\zeta), \dots, x_t + \tau_t(\zeta), y_1 + \sigma_1(\zeta), \dots, y_s + \sigma_s(\zeta))$ ,  
and  $\mathcal{O}_K^{*,U}$  as  $\xi(x_1, \dots, x_t, y_1, \dots, y_s) = (x_1, \dots, x_t, \sigma_1(\xi)y_1, \dots, \sigma_t(\xi)y_t)$

**THEOREM:** (Oeljeklaus-Toma) The OT-manifold  $M := \mathbb{C}^t \times H^s / \Gamma$  is a compact complex manifold, without any non-constant meromorphic functions. When  $t = 1$ , it is locally conformally Kähler. When  $s = 1, t = 1$ , it is an Inoue surface of class  $S^0$ .

**THEOREM:** (Ornea-V.) Let  $M$  be an OT-manifold,  $t = 1$ . Then  $M$  is equipped with a holomorphic 1-dimensional foliation and a transversally Kähler, exact form.

## Stability and transversally Kähler foliations

**THEOREM:** Let  $(M, \Sigma, \omega_0)$  be a compact, complex manifold,  $\dim M > 2$ , and  $\omega_0$  a transversally Kähler, exact form. Let  $\omega_1 := \omega_0 + \theta \wedge I(\theta)$ ,  $\theta \in \Lambda^1(M)$  be a Hermitian form, and  $\omega$  the corresponding Gauduchon form. Consider a vector bundle  $B$  with a Yang-Mills metric,  $\deg_\omega B = 0$ , and let  $\nabla$  denote the Yang-Mills connection. **Then the curvature  $\Theta_B$  of  $\nabla$  satisfies  $\Theta_B(x, \cdot) = 0$  for any  $x \in \Sigma$ .**

**REMARK:** The condition “Then the curvature  $\Theta_B$  of  $\nabla$  satisfies  $\Theta_B(x, \cdot) = 0$  for any  $x \in \ker \omega_0$ ” means that  $(B, \nabla)$  is **locally lifted from the leaf space of the foliation  $\Sigma$ .**

## Leaf space of quasiregular foliations

**REMARK:** When  $\Sigma$  is quasiregular,  $M$  is equipped with a holomorphic projection to the leaf space,  $\pi : M \rightarrow X = M/\Sigma$ . In this situation, the category of coherent sheaves can be described explicitly, in terms of a projective orbifold  $M/\Sigma$ .

**THEOREM:** Let  $F$  be a stable coherent sheaf on a compact, complex manifold  $(M, \Sigma, \omega_0)$ , with a transversally Kähler, exact form,  $\dim M > 2$ . Assume that  $\Sigma$  is quasiregular, and let  $\pi : M \rightarrow X = M/\Sigma$  be the projection map. **Then  $F = \pi^*F_0$ , for some coherent sheaf  $F_0$  on  $M/\Sigma$ .**

**COROLLARY:** In these assumptions, **any coherent sheaf on  $M$  is filtrable**, that is, admits a filtration with rank 1 quotient sheaves.

**REMARK:** Filtrability is a very strong property! **It fails on almost all non-algebraic surfaces.**

## Stability and transversally Kähler foliations (2)

**THEOREM:** Let  $(M, \Sigma, \omega_0)$  be a compact, complex manifold,  $\dim M > 2$ , and  $\omega_0$  a transversally Kähler, exact form. Let  $\omega_1 := \omega_0 - \sqrt{-1} \theta \wedge \bar{\theta}$ ,  $\theta \in \Lambda^{1,0}(M)$  be a Hermitian form, and  $\omega$  the corresponding Gauduchon form. Consider a vector bundle  $B$  with a Yang-Mills metric,  $\deg_\omega B = 0$ , and let  $\nabla$  denote the Yang-Mills connection. **Then the curvature  $\Theta_B$  of  $\nabla$  satisfies  $\Theta_B(x, \cdot) = 0$  for any  $x \in \Sigma$ .**

**Proof:** At a given point  $m \in M$ , let  $\omega_0 = -\sqrt{-1} \sum \theta_i \wedge \bar{\theta}_i$ ,  $\omega = -\sqrt{-1} \sum \theta_i \wedge \bar{\theta}_i - \sqrt{-1} \theta \wedge \bar{\theta}$ , where  $\theta, \theta_1, \dots, \theta_n$  is an orthonormal basis in  $\Lambda^{1,0}(M)$ . Write the curvature of  $B$  as

$$\begin{aligned} \Theta &= \sum_{i \neq j} (\theta_i \wedge \bar{\theta}_j + \bar{\theta}_i \wedge \bar{\theta}_j) \otimes b_{ij} + \sum_i (\theta_i \wedge \bar{\theta}_i) \otimes a_i \\ &\quad + \sum_i (\theta \wedge \bar{\theta}_i + \bar{\theta} \wedge \bar{\theta}_i) \otimes b_i + \theta \wedge \bar{\theta} \otimes a, \end{aligned}$$

with  $b_{ij}, b_i, a_i, a \in \mathfrak{u}(B)$  being skew-Hermitian endomorphisms of  $B$ . Let  $\Xi := \text{Tr}(\Theta \wedge \Theta)$ . Then  $(\sqrt{-1})^n \Xi \wedge \omega_0^{n-2} = \text{Tr}(-\sum b_i^2 + a(\sum a_i))$ . On the other hand,  $\sum a_i + a = \Lambda \Theta = 0$ , hence  $(\sqrt{-1})^n \Xi \wedge \omega_0^{n-2} = \text{Tr}(-\sum b_i^2 - a^2)$ . Since  $\text{Tr}(-a^2)$  is a positive definite form on  $\mathfrak{u}(B)$ , the integral  $\int_M (\sqrt{-1})^n \Xi \wedge \omega_0^{n-2}$  is non-negative, and positive unless  $b_i$  and  $a$  both vanish everywhere. If  $\omega_0$  is exact, this integral vanishes, and  $\Theta(x) = 0$  for any  $x \in \Sigma$ . ■

## Foliation by group orbits

**REMARK:** The condition  $\deg_{\omega} B = 0$  **can be rectified by a tensor multiplication with an appropriate line bundle of non-zero degree.**

**THEOREM:** Let  $(M, \Sigma, \omega_0)$  be a compact, complex manifold,  $\dim M > 2$ , and  $\omega_0$  a transversally Kähler, exact form. Let  $\rho$  denote an action of  $\mathbb{C}$  on  $M$ . Assume that its orbits are leaves of  $\Sigma$ , and that the form  $\omega_0$  is  $\rho$ -invariant. **Then any stable bundle (or coherent sheaf)  $B$  on  $M$  is  $\rho$ -equivariant.**

**Proof:** When  $\deg_{\omega} B = 0$ , the bundle  $B$  is flat on the leaves of  $\Sigma$ , and the parallel transport along leaves is compatible with the connection. When  $\deg_{\omega} B \neq 0$ , we multiply  $B$  by a  $\rho$ -equivariant line bundle  $L$  with non-negative degree. For  $L$  one could take a topologically trivial line bundle with curvature  $const\omega_0$ . ■

## Foliations with dense leave

**COROLLARY:** Let  $(M, \Sigma, \omega_0)$  be a compact, complex manifold,  $\dim M > 2$ , and  $\omega_0$  a transversally Kähler, exact form. Let  $\rho$  denote an action of  $\mathbb{C}$  on  $M$ . Assume that its orbits are leaves of  $\Sigma$ , and that the form  $\omega_0$  is  $\rho$ -invariant. Let  $\pi : M \rightarrow X$  be holomorphic map with general fiber obtained as a closure of  $\rho$ -orbit. **Then any stable coherent sheaf  $F$  on  $M$  with  $\deg_\omega F \neq 0$  is obtained as a pullback of a stable sheaf from  $X$ .**

**Proof:** Let  $G$  be the closure of  $\rho$  in the group of holomorphic isometries of  $M$ . It is a compact Lie group which acts on  $M$ . Consider the action of  $\rho$  on total space of  $F$ ; since  $\rho$  acts on  $\text{Tot } F$  by isometries, its action is extended to  $G$ -action. Since  $F$  is trivial over  $G$ -orbits in  $M$ , it is lifted from the quotient  $M/G$ . ■

**COROLLARY:** In these assumptions, suppose that the orbit of  $\rho$  is dense. **Then all stable coherent sheaves have rank 1.** ■

**QUESTION: What happens for a more general foliation?** For example, on OT-manifold, when there is no such a group action?

**One can build a Hitchin-Simpson theory of characted varieties for such foliated spaces** (the Hitchin-Simpson correspondence **exists already**).