# Deformation theory for holomorphically symplectic manifolds and the proof of Voisin's theorem on deformation of Lagrangian subvarieties

Misha Verbitsky

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**joint work with Nikon Kurnosov**

## Complex manifolds

**DEFINITION:** Let $M$ be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$. **The eigenvalues of this operator are $\pm\sqrt{-1}$.** The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

**DEFINITION:** An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case $I$ is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

**THEOREM: (Newlander-Nirenberg)**
**This definition is equivalent to the standard one.**

**CLAIM: (the Hodge decomposition determines the complex structure)**
Let $M$ be a smooth $2n$-dimensional manifold. **Then there is a bijective correspondence between the set of almost complex structures, and the set of sub-bundles $T^{0,1}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $\dim_{\mathbb{C}} T^{0,1}M = n$ and $T^{0,1}M \cap TM = 0$** (the last condition means that there are no real vectors in $T^{1,0}M$, that is, that $T^{0,1}M \cap T^{1,0}M = 0$).

**Proof:** Set $I\Big|_{T^{1,0}M} = \sqrt{-1}$ and $I\Big|_{T^{0,1}M} = -\sqrt{-1}$. ■

## Hodge theory

**DEFINITION:** Let $(M, I)$ be a complex manifold, $\{U_i\}$ its covering, and and $z_1, ..., z_n$ holomorphic coordinate system on each covering patch. **The bundle $\Lambda^{p,q}(M, I)$ of $(p, q)$-forms on $(M, I)$** is generated locally on each coordinate patch by monomials $dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} \wedge d\overline{z}_{i_{p+1}} \wedge ... \wedge dz_{i_{p+q}}$. **The Hodge decomposition** is a decomposition of vector bundles:

$$\Lambda^d_{\mathbb{C}}(M) = \bigoplus_{p+q=d} \Lambda^{p,q}(M).$$

**DEFINITION:** A manifold is called **Kähler** if it equipped with a closed real (1,1)-form $\omega$ such that $\omega(Ix, x) > 0$ for any non-zero vector $x$.

**THEOREM: ("Hodge decomposition on cohomology")** Let $M$ be a compact Kähler manifold. **Then any cohomology class can be represented as a sum of closed $(p, q)$-forms.**

## Holomorphically symplectic manifolds

**DEFINITION:** Let $(M,I)$ be a complex manifold, and $\Omega \in \Lambda^2(M,\mathbb{C})$ a differential form. We say that $\Omega$ is **non-degenerate** if $\ker \Omega \cap T_{\mathbb{R}}M = 0$. We say that it is **holomorphically symplectic** if it is non-degenerate, $d\Omega = 0$, and $\Omega(IX,Y) = \sqrt{-1}\,\Omega(X,Y)$.

**REMARK:** The equation $\Omega(IX,Y) = \sqrt{-1}\Omega(X,Y)$ means that **$\Omega$ is complex linear with respect to the complex structure on $T_{\mathbb{R}}M$ induced by $I$.**

**REMARK:** Consider the Hodge decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ (decomposition according to eigenvalues of $I$). Since $\Omega(IX,Y) = \sqrt{-1}\,\Omega(X,Y)$ and $I(Z) = -\sqrt{-1}\,Z$ for any $Z \in T^{0,1}(M)$, we have $\ker(\Omega) \supset T^{0,1}(M)$. Since $\ker \Omega \cap T_{\mathbb{R}}M = 0$, real dimension of its kernel is at most $\dim_{\mathbb{R}} M$, giving $\dim_{\mathbb{R}} \ker \Omega = \dim M$. **Therefore,** $\ker(\Omega) = T^{0,1}M$**.**

**COROLLARY:** Let $\Omega$ be a holomorphically symplectic form on a complex manifold $(M,I)$. **Then $I$ is determined by $\Omega$ uniquely.**

## C-symplectic structures

**DEFINITION: (Bogomolov, Deev, V.)** Let $M$ be a smooth $4n$-dimensional manifold. A complex-valued form $\Omega$ on $M$ is called **almost C-symplectic** if $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is a non-degenerate volume form. It is called **C-symplectic** when it is also closed.

**THEOREM:** Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a C-symplectic form, and $T^{0,1}_\Omega(M)$ be equal to $\ker \Omega$, where $\ker \Omega := \{v \in TM \otimes \mathbb{C} \ \ | \ \ \Omega \lrcorner v = 0\}$. Then $T^{0,1}_\Omega(M) \oplus \overline{T^{0,1}_\Omega(M)} = TM \otimes_{\mathbb{R}} \mathbb{C}$, hence **the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure $I_\Omega$ on $M$.** If, in addition, **$\Omega$ is closed, $I_\Omega$ is integrable, and $\Omega$ is holomorphically symplectic on $(M, I_\Omega)$.**

**Proof:** Rank of $\Omega$ is $2n$ because $\Omega^{n+1} = 0$ and $\operatorname{Re} \Omega$ is non-degenerate. Then $\ker \Omega \oplus \overline{\ker \Omega} = T_{\mathbb{C}} M$. The relation $[T^{0,1}_\Omega(M), T^{0,1}_\Omega(M)] \subset T^{0,1}_\Omega(M)$ follows from Cartan's formula

$$d\Omega(X_1, X_2, X_3) = \frac{1}{6} \sum_{\sigma \in \Sigma_3} (-1)^{\tilde{\sigma}} \operatorname{Lie}_{X_{\sigma_1}} \Omega(X_{\sigma_2}, X_{\sigma_3}) + (-1)^{\tilde{\sigma}} \Omega([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3})$$

which gives, for all $X, Y \in T^{0,1} M$, and any $Z \in TM$,

$$d\Omega(X, Y, Z) = \Omega([X, Y], Z),$$

implying that $[X, Y] \in T^{0,1} M$. ■

## Local Torelli theorem

**DEFINITION:** Let $(M, I, \Omega)$ be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient $\mathrm{CTeich} := \frac{\mathrm{CSymp}}{\mathrm{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map $\mathrm{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking $(M, I, \Omega)$ to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

**DEFINITION:** Let $M$ be a compact complex manifold. We say that $M$ **satisfies $\partial\bar\partial$-lemma in term $\Lambda^{p,q}(M)$** if any $\partial$-closed, $\bar\partial$-exact $(p,q)$-form belongs to the image of $\partial\bar\partial$.

**THEOREM: ("Local Torelli theorem"; Kurnosov, V.)**
Let $(M, \Omega)$ be a C-symplectic manifold. Assume that $H^{0,1}(M) = 0$, $H^{2,0}(M) = \mathbb{C}$ and $M$ satisfies $\partial\bar\partial$-lemma in $\Lambda^{1,2}(M)$ and has Hodge decomposition in $H^2(M)$. Let $W := \frac{H^2(M,\mathbb{C})}{\langle\overline{\Omega}\rangle}$. Then the period map composed with the natural projection $H^2(M, \mathbb{C}) \mapsto W$ **defines a local difeomorphism from CTeich to a neighbourhood of 0 in $W$.**

**REMARK:** Today I will not give the proof of this theorem, but I will explain an explicit construction of a local deformation which is mapped to a neighbourhood of 0 in $W$ diffeomorphically.

## Schouten brackets

**DEFINITION:** Let $M$ be a complex manifold, and $\Lambda^{0,p}(M) \otimes T^{1,0}M$ the sheaf of $T^{1,0}M$-valued $(0,p)$-forms. Consider the commutator bracket $[\cdot,\cdot]$ on $T^{1,0}M$, and let $\overline{\mathcal{O}}_M$ denote the sheaf of antiholomorphic functions. Since $[\cdot,\cdot]$ is $\overline{\mathcal{O}}_M$-linear, it is naturally extended to $\Lambda^{0,p}(M)\otimes_{C^\infty M}T^{1,0}M = \overline{\Omega^p M}\otimes_{\overline{\mathcal{O}}_M}T^{1,0}M$, giving a bracket

$$[\cdot,\cdot]: \ \Lambda^{0,p}(M)\otimes T^{1,0}M \times \Lambda^{0,q}(M)\otimes T^{1,0}M \longrightarrow \Lambda^{0,p+q}(M)\otimes T^{1,0}M.$$

This bracket is called **Schouten bracket**.

**REMARK:** Since $[\cdot,\cdot]$ is $\overline{\mathcal{O}}_M$-linear, the Schouten bracket satisfies the Leibnitz identity:

$$\overline{\partial}([\alpha,\beta]) = [\overline{\partial}\alpha,\beta] + [\alpha,\overline{\partial}\beta].$$

**This allows one to extend the Schouten bracket to the $\overline{\partial}$-cohomology of the complex $(\Lambda^{0,*}(M)\otimes T^{1,0}M, \overline{\partial})$,** which coincide with the cohomology of the sheaf of holomorphic vector fields: $[\cdot,\cdot]: \ H^p(TM)\times H^q(TM) \longrightarrow H^{p+q}(TM)$.

## Maurer-Cartan equation and deformations

**CLAIM:** Let $(M, I)$ be an almost complex manifold, and $B$ an abstract vector bundle over $\mathbb{C}$ isomorphic to $\Lambda^{0,1}(M)$. Consider a differential operator $\overline{\partial}: C^\infty M \longrightarrow B = \Lambda^{0,1}(M)$ satisfying the Leibnitz rule. Its symbol is a linear map $u: \Lambda^1(M, \mathbb{C}) \longrightarrow B$. Then $B = \frac{\Lambda^1(M,\mathbb{C})}{\ker u} = \Lambda^{0,1}(M)$. Extend $\overline{\partial}: C^\infty M \longrightarrow B$ to the corresponding exterior algebra using the Leibnitz rule:

$$C^\infty M \xrightarrow{\overline{\partial}} B \xrightarrow{\overline{\partial}} \Lambda^2 B \xrightarrow{\overline{\partial}} \Lambda^3 B \xrightarrow{\overline{\partial}} ...$$

**Then integrability of $I$ is equivalent to $\overline{\partial}^2 = 0$.**
**Proof:** This is essentially the Newlander-Nirenberg theorem. ■

**REMARK:** Almost complex deformations of $I$ are given by the sections $\gamma \in T^{1,0}M \otimes \Lambda^{0,1}(M)$, with the integrability relation $(\overline{\partial} + \gamma)^2 = 0$ rewritten as **the Maurer-Cartan equation** $\overline{\partial}(\gamma) = -\{\gamma, \gamma\}$. Here $\overline{\partial}(\gamma)$ is identified with the anticommutator $\{\overline{\partial}, \gamma\}$, and $\{\gamma, \gamma\}$ is anticommutator of $\gamma$ with itself, where $\gamma$ is considered as a $\Lambda^{0,1}(M)$-valued differential operator. **This identifies $\{\gamma, \gamma\}$ with the Schouten bracket.**

**REMARK:** We shall write $[\gamma, \gamma]$ instead of $\{\gamma, \gamma\}$, because this usage is more common.

## Solving the Maurer-Cartan equation recursively

**DEFINITION: The Kuranishi deformation space,** can be defined as **the space of solutions of Maurer-Cartan equation $\overline{\partial}(\gamma) = -[\gamma, \gamma]$ modulo the diffeomorphism action.**

**DEFINITION:** Write $\gamma$ as power series, $\gamma = \sum_{i=0}^{\infty} t^{i+1}\gamma_i$. **Then the Maurer-Cartan becomes**

$$\overline{\partial}\gamma_0 = 0, \quad \overline{\partial}\gamma_p = -\sum_{i+j=p-1} [\gamma_i, \gamma_j]. \quad (**)$$

We say that deformations of complex structures are **unobstructed** if the solutions $\gamma_1, ..., \gamma_n, ...$ of (**) can be found for $\gamma_0$ in any given cohomology class $[\gamma_0] \in H^1(M, TM)$.

**REMARK 1:** Notice that **the sum $\sum_{i+j=p-1}[\gamma_i, \gamma_j]$ is always $\overline{\partial}$-closed.** Indeed, the Schouten bracket commutes with $\overline{\partial}$, hence

$$\overline{\partial} \sum_{i+j=p-1} [\gamma_i, \gamma_j] = -\sum_{i+j+k=p-1} [\gamma_i, [\gamma_j, \gamma_k]] + [[\gamma_i, \gamma_j], \gamma_k]. \quad (***)$$

vanishes as a sum of triple supercommutators. **Obstructions to deformations** are given by cohomology classes of the sums $\sum_{i+j=p-1}[\gamma_i, \gamma_j]$, which are defined inductively. These classes are called **Massey powers** of $\gamma_0$.

## Tian-Todorov lemma

**DEFINITION:** Assume that $M$ is a complex $n$-manifold with trivial canonical bundle $K_M$, and $\Phi$ a non-degenerate section of $K_M$. We call a pair $(M,\Phi)$ **a Calabi-Yau manifold**. Substitution of a vector field into $\Phi$ gives an isomorphism $TM \cong \Omega^{n-1}(M)$. Similarly, one obtains an isomorphism

$$\Lambda^{0,q}M \otimes \Lambda^p TM \longrightarrow \Lambda^{0,q}M \otimes \Lambda^{n-p,0}M = \Lambda^{n-q,p}M. \qquad (*)$$

**Yukawa product** $\bullet$ : $\Lambda^{p,q}M \otimes \Lambda^{p_1,q_1}M \longrightarrow \Lambda^{p+p_1-n,q+q_1}M$ is obtained from the usual product

$$\Lambda^{0,q}M \otimes \Lambda^p TM \times \Lambda^{0,q_1}M \otimes \Lambda^{p_1}TM \longrightarrow \Lambda^{0,q+q_1}M \otimes \Lambda^{p+p_1}TM$$

using the isomorphism (*).

**TIAN-TODOROV LEMMA:** Let $(M,\Phi)$ be a Calabi-Yau manifold, and

$$[\cdot,\cdot]: \Lambda^{0,p}(M) \otimes T^{1,0}M \times \Lambda^{0,q}(M) \otimes T^{1,0}M \longrightarrow \Lambda^{0,p+q}(M) \otimes T^{1,0}M.$$

its Schouten bracket. Using the isomorphism (*), we can interpret Schouten bracket as a map

$$[\cdot,\cdot]: \Lambda^{n-1,p}(M) \times \Lambda^{n-1,q}(M) \longrightarrow \Lambda^{n-1,p+q}(M).$$

**Then, for any $\alpha \in \Lambda^{n-1,p}(M)$, $\beta \in \Lambda^{n-1,p_1}(M)$, one has**

$$[\alpha,\beta] = \partial(\alpha \bullet \beta) - (\partial\alpha) \bullet \beta - (-1)^{n-1+p} \alpha \bullet (\partial\beta),$$

where $\bullet$ denotes the Yukawa product.

## $dd^c$-lemma

**DEFINITION:** Let $M$ be a complex manifold, and $I: TM \longrightarrow TM$ its complex structure operator. **The twisted differential** of $M$ is $IdI^{-1}: \Lambda^*(M) \longrightarrow \Lambda^{*+1}(M)$, where $I$ acts on 1-forms as an operator dual to $I: TM \longrightarrow TM$, and on the rest of differential forms multiplicatively.

**REMARK:** Consider the Hodge decomposition of the de Rham differential, $d = \partial + \overline{\partial}$, where $\partial: \Lambda^{p,q}(M,I) \longrightarrow \Lambda^{p+1,q}(M,I)$ and $\overline{\partial}: \Lambda^{p,q}(M,I) \longrightarrow \Lambda^{p+1,q}(M,I)$. **Then $d = \operatorname{Re}\partial$ and $d^c = \operatorname{Im}\partial$.** Also, $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

**THEOREM:** (**$dd^c$-lemma**) Let $\eta$ be a form on a compact Kähler manifold, satisfying one of the following conditions.
(1). $\eta$ is an exact $(p,q)$-form. (2). $\eta$ is $d$-exact, $d^c$-closed.
**Then $\eta$ is $dd^c$-exact,** that is, $\eta \in \operatorname{im} dd^c$. Equivalently, **if $\eta$ is $\partial$-exact and $\overline{\partial}$-closed, it is $dd^c$-exact.**

**REMARK:** This statement is weaker that the Kähler condition, but it immediately implies almost every cohomological property of Kähler manifolds, except the Lefschetz $\mathfrak{sl}(2)$-action. In particular, **$dd^c$-lemma is sufficient to prove the Bogomolov-Tian-Todorov theorem**, claiming that the deformations of Calabi-Yau manifolds are unobstructed.

# Bogomolov-Tian-Todorov theorem

**THEOREM:** Let $M$ be a compact complex $n$-manifold with trivial canonical bundle which satisfies $dd^c$-lemma. **Then its deformations are unobstructed**.

**Proof. Step 1:** Let's start with a cohomology class $[\gamma_0] \in H^1(TM) = H^1(\Omega^{n-1}M)$. To prove that the deformations are unobstructed, we need to solve the equation system

$$\overline{\partial}\gamma_0 = 0, \quad \overline{\partial}\gamma_p = -\sum_{i+j=p-1} [\gamma_i, \gamma_j]. \quad (**)$$

recursively, starting from a representative $\gamma_0$ of $[\gamma_0]$. Identifying $\Lambda^{0,1}(T^{1,0}M)$ with $\Lambda^{0,1}(\Lambda^{n-1,0}M) = \Lambda^{n-1,1}(M)$, **we choose a representative $\gamma_0 \in \Lambda^{n-1,1}(M)$ of $[\gamma_0]$ which is $\partial$ and $\overline{\partial}$-closed;** this is possible to do using $\partial\overline{\partial}$-lemma (in Kähler situation, take a harmonic representative).

**Step 2:** Using induction, we may assume that (**) is solved up to $\gamma_{n-1}$, and, moreover, the solutions satisfy $\partial\gamma_i = 0$. By Tian-Todorov lemma,

$$\alpha := [\gamma_i, \gamma_j] = \partial(\gamma_i \bullet \gamma_j) - (\partial\gamma_i) \bullet \gamma_j - (-1)^{n-1+p}\gamma_i \bullet (\partial\gamma_j) = \partial(\gamma_i \bullet \gamma_j),$$

hence it is $\partial$-exact; as shown in Remark 1 above, it is also $\overline{\partial}$-closed. By $dd^c$-lemma, $\alpha$ is $\partial\overline{\partial}$-exact. This implies that $-\sum_{i+j=n-1}[\gamma_i, \gamma_j] = \overline{\partial}\partial\beta$. **Taking $\gamma_n := \partial\beta$, we obtain a solution of (**) which is also $\partial$-closed, hence satisfy the induction assumptions.** ■

## Tian-Todorov lemma for holomorphically symplectic manifolds

Let now $\Omega$ be a holomorphically symplectic form on a complex manifold $M$, $\dim_{\mathbb{C}} M = 2n$. Then $TM \cong \Omega^1 M$, hence the Schouten bracket is defined as

$$\Lambda^{1,p}(M) \times \Lambda^{1,q}(M) \longrightarrow \Lambda^{1,p+q}(M).$$

**LEMMA:** Let $M$ be a holomorphic symplectic manifold. Consider the operators $L_\Omega(\alpha) := \Omega \wedge \alpha$, $H_\Omega$ acting as multiplication by $n-p$ on $\Lambda^{p,q}(M)$, and $\Lambda_\Omega := \star\Lambda\star$. **Then $L_\Omega, H_\Omega, \Lambda_\Omega$ satisfy the $\mathfrak{sl}(2)$ relations, similar to the Lefschetz triple: $[H_\Omega, L_\Omega] = 2L_\Omega, \quad [H_\Omega, \Lambda_\Omega] = -2\Lambda_\Omega, [L_\Omega, \Lambda_\Omega] = H_\Omega$.** ■

**LEMMA: (Tian-Todorov for holomorphically symplectic manifolds)**
Let $(M, \Omega)$ be a holomorphically symplectic manifold, and

$$[\cdot,\cdot]_\Omega : \ \Lambda^{1,p}(M) \times \Lambda^{1,q}(M) \longrightarrow \Lambda^{1,p+q}(M).$$

the Schouten bracket. **Then for any $a, b \in \Lambda^{1,*}(M)$, one has**

$$[a,b] = \delta(a \wedge b) - (\delta a) \wedge b - (-1)^{\tilde{a}} a \wedge \delta(b),$$

**where $\tilde{a}$ is parity of $a$, and $\delta := [\Lambda_\Omega, \partial]$.**

**Proof:** Same as for the usual Tian-Todorov. ■

## Maurer-Cartan for Hamiltonian vector fields

**REMARK:** A solution of the Maurer-Cartan equation $\left(\overline{\partial} + \sum_{i=0}^{\infty} t^{i+1}\gamma_i\right)^2 = 0$ **gives a holomorphically symplectic deformation whenever all $\gamma_i$ belong to $\Lambda^{0,1}(M) \otimes \mathcal{H}\mathrm{am}_M$.** Here $t$ is a formal parameter, or $t$ is chosem in such a way that this sum converges.

Using $\Omega$ to identify vector fields and 1-forms, the sheaf of Hamiltonian vector fields can be embedded to $\Lambda^{1,0}(M)$ as a sheaf of $\partial$-closed (1,0)-forms.

Similarly, if we use $\Omega$ to consider $\gamma_i$ as sections of $\Lambda^{0,1}(M) \otimes T^{1,0}M = \Lambda^{1,1}(M)$, the condition $\gamma_i \in \Lambda^{0,1}(M) \otimes \mathcal{H}\mathrm{am}_M$ is interpreted as $\partial\gamma_i = 0$.

**DEFINITION:** Let $(M, I, \Omega)$ be a holomorphically symplectic manifold. We say that the **holomorphic symplectic deformations of $(M, I, \Omega)$ are unobstructed** if for any $\overline{\partial}$- and $\partial$-closed $\gamma_0 \in \Lambda^{1,1}(M)$ the Maurer-Cartan equation

$$\overline{\partial}\gamma_p = -\sum_{i+j=p-1} [\gamma_i, \gamma_j], \quad p = 1, 2, 3, ...$$

has a solution $(\gamma_1, \gamma_2, ...,)$, with $\gamma_i \in \Lambda^{1,1}(M)$ $\partial$-closed.

## Almost C-symplectic forms

The proof of local Torelli for K3 involves solving an order 2 equation $\Omega\wedge\rho^{0,2} = -\rho^{1,1}\wedge\rho^{1,1}$, because the condition $\Omega^2 = 0$ is quadratic. To wpork any dimension **we write a degree 2 polynomial equation which describes almost C-symplectic structures in the space of all complex-valued 2-forms.**

**DEFINITION:** Let $V$ be a real vector space of dimension $4n$, and $\Lambda^2_{\mathbb C} V := \Lambda^2 V\otimes_{\mathbb R}\mathbb C$. A 2-form $\Omega\in\Lambda^2_{\mathbb C} V$ is **C-symplectic** if $\Omega^n\wedge\overline\Omega^n\neq 0$ and $\Omega^{n+1}=0$.

**Claim 1:** Fix a complex structure $I\in \operatorname{End}_{\mathbb R} V, I^2=-\operatorname{Id}$ on $V$, and let $\Theta\in\Lambda^2_{\mathbb C} V$ be a C-symplectic form. Denote the $(2,0)$-component of $\Theta$ by $\Omega$, let $\eta^{1,1}$ be its $(1,1)$-component and $\eta^{0,2}$ be its $(0,2)$-component. Assume that $\Omega$ is C-symplectic, that is, has maximal rank. **Then**

$$\eta^{1,1}\wedge\eta^{1,1}\wedge\Omega^{n-1} = -\eta^{0,2}\wedge\Omega^n. \quad (*)$$

Moreover, **for any (1,1)-form $\eta^{1,1}$, there exists a unique $(0,2)$-form $\eta^{0,2}$ such that (*) holds.**

**Proof. Step 1:** The $(2n,2)$-component of $\Theta^{n+1}$ is equal to $\eta^{1,1}\wedge\eta^{1,1}\wedge\Omega^{n-2}+\eta^{0,2}\wedge\Omega^n$; now, $\Theta^{n+1}=0$ implies (*) immediately.

**Step 2:** The map $\Lambda^{0,2}V \xrightarrow{\wedge\Omega^n} \Lambda^{2n,2}V$ is clearly an isomorphism. Existence and uniqueness of $\eta^{0,2}$ solving (*) follows from this observation. ■

## The operator $\Lambda_\Omega$

We obtained a quadratic equation (*) from an order $n+1$-equation $\Theta^{n+1} = 0$. **We are going to show that the converse is also true:** (*) implies $\Theta^{n+1} = 0$, at least in a neighbourxood of a C-symplectic structure $\Omega \in \Lambda^{2,0}_{\mathbb{C}} V$.

**DEFINITION:** Fix a complex structure $I \in \operatorname{End}_{\mathbb{R}} V, I^2 = -\operatorname{Id}$ on $V$, and let $\Omega \in \Lambda^{2,0}_{\mathbb{C}} V$ be a C-symplectic form. Consider a (2,2)-form $\Theta \in \Lambda^{2,2} V$ and let $u \in \Lambda^{0,2} V$ be a $(0,2)$-form which satisfies $u \wedge \Omega^n = \Theta \wedge \Omega^{n-1}$. By Step 2 of Claim 1, such $u$ exists for any (2,2)-form $\Theta$. **The map which takes $\Theta$ to $u$ is denoted $\Theta \mapsto \Lambda_\Omega \Theta$.**

**REMARK:** Using this notation, **the equation (*) can be written as $\eta^{0,2} = -\Lambda_\Omega(\eta^{1,1} \wedge \eta^{1,1})$.**

**THEOREM A:** Let $V$ be a real vector space of dimension $4n$, and $I \in \operatorname{End}_{\mathbb{R}} V, I^2 = -\operatorname{Id}$ a complex structure. Denote by $Z$ the space of C-symplectic structures $\Theta \in \Lambda^2_{\mathbb{C}} V$, such that $\Theta^{2,0}$ is non-degenerate, and let $Z_1$ be the space of all triples $\Theta = \Omega + \eta^{1,1} + \eta^{0,2}$, where $\Omega$ is a non-degenerate (2,0)-form, and $\eta^{0,2} = -\Lambda_\Omega(\eta^{1,1} \wedge \eta^{1,1})$. **Then $Z = Z_1$ in a sufficiently small neighbourhood of a given non-degenerate (2,0)-form $\Theta_0$.**

**This theorem will not be proven today.**

## The local Torelli theorem for C-symplectic manifolds

**COROLLARY:** Let $(M, I, \Omega)$ be a compact holomorphically symplectic manifold which satisfies $\partial\bar\partial$-lemma in term $\Lambda^{2,1}(M)$, and $\eta_0$ a closed $(1,1)$-form. Consider a family of solutions of the Maurer-Catran equation

$$\partial\eta_n = 0, \quad \bar\partial\eta_n = \sum_{i+j=n-1} \partial(\Lambda_\Omega(\eta_i \wedge \eta_j)). \quad (***)$$

which exists by holomorphic symplectic Bogomolov-Tian-Todorv lemma, and let $\eta := \sum t^{i+1}\eta_i$. **Then $\Omega_\eta := \Omega + \eta - \Lambda_\Omega(\eta \wedge \eta)$ gives a formal deformation of C-symplectic structures,** which **can be chosen convergent for $t$ sufficiently small and an appropriate choice of solutions $\eta_i$.**

**Proof:** By Theorem A, $\Omega_\eta$ is an almost C-symplectic structure. It is closed, which follows from (***) immediately. Convergence of $\sum t^{i+1}\eta_i$ follows from a routine calculation because the operator $\bar\partial^{-1} = \bar\partial^* \Delta_{\bar\partial}^{-1}$ which is used in solving (***) is compact, and the Green operator $\Delta_{\bar\partial}^{-1}$ is a compact Hermitian operator. ■

## Holomorphic Lagrangian subvarieties

**DEFINITION:** Let $(M, \Omega)$ be a holomorphically symplectic manifold, and $X \subset (M, \Omega)$ a complex subvariety. It is called **holomorphic Lagrangian** if $\Omega$ restricted to the set of smooth points of $X$ vanishes.

**PROPOSITION: (Hitchin's lemma)**
Let $X \subset M$ be a real submanifold (or closed real analytic subvariety) such that $\Omega|_X = 0$ and $\dim_{\mathbb R} X = \frac{1}{2}\dim_{\mathbb R} M$. **Then $X$ is a complex subvariety.**

**Proof. Step 1:** This statement would follow if we prove the following linear-algebraic statement. Let $(V, \Omega)$ be a real vector space equipped with a C-symplectic form, $I: V \longrightarrow V$ the induced complex structure operator, and $W \subset V$ a real subspace such that $\dim_{\mathbb R} W = \frac{1}{2}\dim_{\mathbb R} V$ and $\Omega|_W = 0$. **Then $I(W) = W$**, that is, **$W$ is a complex subspace of $V$.**

**Proof:** Let $u, w \in W$. Since $\Omega$ is $I$-linear, one has $0 = \sqrt{-1}\Omega(u, w) = \Omega(Iu, w)$, hence the space $W_u := \langle W + Iu\rangle$ generated by $W$ and $Iu$ is Lagrangian with respect to $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$. Since the forms $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$ are non-degenerate, dimension of $W_u$ cannot be bigger than $\dim_{\mathbb R} W = \frac{1}{2}\dim_{\mathbb R} V$, hence $W_u = W$.
■

## "Good" subvarieties

**DEFINITION:** Consider a closed complex subvariety $X \subset M$, and let $\tilde M$ a blow-up of $M$ such that the proper preimage of $X$ is a subvariety $\tilde X \subset \tilde M$ which has simple normal crossings. The **essential skeleton** of $X$ is a CW-complex associated with $\tilde X$ as follows: its vertices are irreducible components of $\tilde X$, and its $k$-simplexes with vertices associated to the components $X_1, ..., X_k$ are irreducible components of the intersection $\bigcup_{i=1}^k X_k$. By a theorem of D. A. Stepanov, **the homotopy type of the essential skeleton is independent from the choice of resolution.**

**DEFINITION:** We call a function $f$ on $X$ **smooth** if its pullback to $\tilde X$ is smooth.

**DEFINITION:** A closed, compact subvariety $X \subset M$ is called **good** if the resolution of $X$ is Kähler, any smooth function on $X$ can be extended to a smooth function in a neighbourhood of $X$ in $M$, and the essential skeleton $S$ of $X$ satisfies $H^1(S) = 0$.

## $dd^c$-lemma and essential skeleton

**Proposition 4:** Let $\eta$ be a (1,1)-form on $M$ which is exact on $X \subset M$, which is a "good" complex subvariety. **Then $\eta = dd^c f$ in a neighbourhood of $X$.**

**Proof. Step 1:** Let $\tilde{X}_0$ be the resolution of $X$, obtained from $\tilde{X}$ by taking apart the branches. **Then the pullback $\eta_0$ of $\eta$ to $\tilde{X}_0$ is $dd^c$-exact,** because $\tilde{X}_0$ is smooth and Kähler, hence $\eta_0 = dd^c f$.

**Step 2:** Let $\pi : \tilde{X}_0 \longrightarrow X$ be the projection, and $x \in X$ any point. Since $\ker dd^c$ is holomorphic plus antiholomorphic functions, on a compact complex variety $\ker dd^c$ is constant functions. On each irreducible component of $\pi^{-1}(x)$, the function $f$ satisfies $dd^c f = 0$, hence it is constant. Therefore, the function $f$ such that $\eta_0 = dd^c f$ is uniquely, up to a constant, defined on each connected component $X_i$ of $\tilde{X}_0$. To show that $f$ is a pullback of a function on $\tilde{X}$, and hence on $X$, **we need to chose these constants in such a way that $f|_{X_i}$ agrees on all intersections $X_i \cap X_j$.**

**Step 3:** Choose a function $f_i$ which satisfies $dd^c f_i = \eta_0|_{X_i}$ on each of these components. The difference $f_i|_{X_i \cap X_j} - f_j|_{X_i \cap X_j}$ is a constant function on each intersection $X_i \cap X_j$ which sums up to zero on triple intersections, hence it defines a 1-cocycle on $S$. **To choose $f_i$ which agree on intersections, we need to show that this cocycle is exact;** this can be ensured by assuming that $H^1(S) = 0$, where $S$ is the essential skeleton. ■

## Deformations of Lagrangian subvarieties

**THEOREM: (joint with N. Kurnosov)**
Let $(M, \Omega)$ be a compact C-symplectic manifold satisfying the assumptions of local Torelli theorem, $X \subset (M, \Omega)$ a good closed holomorphic Lagrangian subvariety. Consider the space $\operatorname{CTeich}_X \subset \operatorname{CTeich}$ consisting of all $\Omega' \in \operatorname{CTeich}$ such that the restriction of $\Omega'$ to $X$ is exact. Assume that $X$ is "good" in the sense of the above definition. Then locally around $\Omega \in \operatorname{CTeich}_X$ **there exist a choice of holomorphic symplectic representatives $\Omega_t$, smoothly depending on $t \in \operatorname{CTeich}_X$, such that $\Omega_t|_X = 0$ for all $t$.**

**Proof:** Next slide.

**REMARK:** In other word, for a sufficiently small deformation $\Omega_t \in \operatorname{CTeich}_X$ of the C-symplectic structure $\Omega$ in $\operatorname{CTeich}_X$, **the variety $X$ can be deformed to a Lagrangian subvariety in $(M, \Omega_t)$.**

**REMARK:** This result was proven by Voisin for smooth holomorphic Lagrangian $X$ in projective $M$, and by C. Lehn when $X$ are SNC holomorphic Lagrangian subvarieties in projective $M$. **We needed this result for Bogomolov-Guan manifolds**, and found **an improved proof of Voisin's theorem which also works for singular $X$ and non-Kähler $M$.**

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**Proof. Step 1:** After rescaling, we may assume that $[\Omega_t]^{2,0} = \Omega$. We write $\Omega_t$ by solving (*) recursively, starting with a closed $(1,1)$-form $\gamma_0$ representing $[\Omega_t]^{1,1}$:

$$\bar\partial \gamma_n = \partial \Lambda_\Omega \left( \sum_{i+j=n-1} \gamma_i \wedge \gamma_j \right). \quad (**)$$

Adding $dd^c f_n$ to $\gamma_n$ won't affect (**). I will show that **we can always choose $\gamma_n$ such that $\gamma_n|_X = 0$ if $\gamma_i|_X = 0$ for all $i < n$.**

## Deformations of Lagrangian subvarieties (2)

**Step 2:** We use induction in $n$. Since $X$ is "good", it satisfies the $dd^c$-lemma. By Proposition 4, we can replace $\gamma_0$ by $\gamma_0 - dd^c f$ such that $dd^c f|_X = \gamma_0|_X$. Smoothly extending $f$ to a neighbourhood of $X$ and replacing $\gamma_0$ by $\gamma_0 - dd^c f$, we obtain another closed representative of $[\gamma_0]$ which satisfies $\gamma_0|_X = 0$. **This is the basis of induction.**

**Step 3:** We need the following linear-algebraic lemma. Let $X \subset (M, \Omega)$ be a holomorphic Lagrangian subvariety, and $\alpha_1, \alpha_2 \in \Lambda^{1,1}(M)$ (1,1)-forms which satisfty $\alpha_1|_X = \alpha_2|_X = 0$. **Then $\Lambda_\Omega(\alpha_1 \wedge \alpha_2)|_X = 0$.** This is a local statement; using Darboux theorem, we introduce the coordinates such that $\Omega = \sum_i dp_i \wedge dq_i$ and all $q_i$ are constant on $X$. Then $\alpha_k = \sum_{i,j} a_{ijk} dp_i \wedge d\overline{q}_j + \sum_{i,j} b_{ijk} dq_i \wedge d\overline{p}_j + \sum_{i,j} c_{ijk} dq_i \wedge d\overline{q}_j$, which gives

$$\Lambda_\Omega(\alpha_1 \wedge \alpha_2) = -\sum_{i,j,j'} a_{ijk} b_{ij'k} d\overline{q}_j \wedge d\overline{p}_{j'} - \sum_{i,j,j'} a_{ijk} c_{ij'k} d\overline{q}_j \wedge d\overline{q}_{j'}.$$

**This form vanishes on $X$ because $\overline{q}_j$ is constant on $X$.**

**Step 4:** Suppose that $\gamma_i|_X = 0$ for all $i < n$. Then $u := \Lambda_\Omega \left(\sum_{i+j=n-1} \gamma_i \wedge \gamma_j\right)|_X = 0$ (Step 3). Solving the equation $\overline{\partial}\gamma_n = \partial u$, $\gamma_n \in \operatorname{im} \partial$, we obtain a solution $\gamma_n$ which is $\partial$-exact and $\overline{\partial}$-closed on $X$. Applying Proposition 4 again, **we replace $\gamma_n$ by another solution $\gamma_n - dd^c f$ which vanishes on $X$.** ■