Local Torelli theorem for C-symplectic structures

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Teichmüller space for symplectic structures

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ **is a vector space, and** Symp **an infinite-dimensional (Fréchet) manifold.**

DEFINITION: Let Diff_0 be the group of isotopies of M, that is, **the connected component of the diffeomorphism group. Teichmüller space of symplectic structures on M** is defined as the quotient space $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$.

REMARK: Let $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ be the mapping class group of M. The quotient $\operatorname{Teich}_s / \Gamma = \operatorname{Symp} / \operatorname{Diff}$ **is identified with the set of symplectic structures up to diffeomorphism.**

Moser's theorem

DEFINITION: Let M be compact. Define **the period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

Moser's lemma: Let ω_t, $t \in [0,1]$ be a smooth family of symplectic structures on a compact manifold M. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are diffeomorphic.**

Proof of Moser's theorem: The period map $P: U \longrightarrow H^2(M, \mathbb{R})$ is a smooth submersion of infinite-dimensional smooth manifolds. By Moser's lemma, the fibers of P are 0-dimensional. **Therefore, P is locally a diffeomorphism.** ■

The proof of Moser's lemma

Moser's lemma: Let ω_t, $t \in [0,1]$ be a smooth family of symplectic structures on a compact manifold M. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then there exists a smooth family $\Psi_t \in \operatorname{Diff}_0(M)$ of diffeomorphisms such that $\Psi_t^* \omega_0 = \omega_t$.**

Proof: We construct Ψ_t as a solution of the equation $\frac{d\Psi_t}{dt} = X_t$, where $X_t \in TM$ is a vector field depending on $t \in [0,1]$.

Step 1: Since all ω_t are cohomologous, the form $\frac{d\omega_t}{dt}$ is exact. This gives $\frac{d\omega_t}{dt} = d\eta_t$, where $\eta_t \in \Lambda^1(M)$ smoothly depends on $t \in [0,1]$. Let X_t be the vector field which satisfies $\omega_t \lrcorner X_t = \eta_t$. **Cartan's formula gives** $\operatorname{Lie}_{X_t} \omega_t = d(\omega_t \lrcorner X_t) = d\eta_t = \frac{d\omega_t}{dt}$.

Step 2: Let Ψ_t be the flow of diffeomorphisms obtained by integrating X_t. By construction, $\operatorname{Lie}_{X_t} \omega_t = \frac{d\omega_t}{dt}$. Integrating it in t, we obtain

$$\Psi_1^* \omega_0 = \int_0^1 \Psi_t \operatorname{Lie}_{X_t} \omega_t dt = \int_0^1 \frac{d\omega_t}{dt} dt = \omega_1.$$

■

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$. **The eigenvalues of this operator are $\pm\sqrt{-1}$.** The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the standard one.

CLAIM: (the Hodge decomposition determines the complex structure)
Let M be a smooth $2n$-dimensional manifold. **Then there is a bijective correspondence between the set of almost complex structures, and the set of sub-bundles $T^{0,1}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $\dim_{\mathbb{C}} T^{0,1}M = n$ and $T^{0,1}M \cap TM = 0$** (the last condition means that there are no real vectors in $T^{1,0}M$, that is, that $T^{0,1}M \cap T^{1,0}M = 0$).

Proof: Set $I\Big|_{T^{1,0}M} = \sqrt{-1}$ and $I\Big|_{T^{0,1}M} = -\sqrt{-1}$. ■

Hodge theory

DEFINITION: Let (M,I) be a complex manifold, $\{U_i\}$ its covering, and and $z_1, ..., z_n$ holomorphic coordinate system on each covering patch. **The bundle $\Lambda^{p,q}(M,I)$ of (p,q)-forms on (M,I)** is generated locally on each coordinate patch by monomials $dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} \wedge d\overline{z}_{i_{p+1}} \wedge ... \wedge dz_{i_{p+q}}$. **The Hodge decomposition** is a decomposition of vector bundles:

$$\Lambda^d_{\mathbb{C}}(M) = \bigoplus_{p+q=d} \Lambda^{p,q}(M).$$

DEFINITION: A manifold is called **Kähler** if it equipped with a closed real (1,1)-form ω such that $\omega(Ix,x) > 0$ for any non-zero vector x.

THEOREM: ("Hodge decomposition on cohomology") Let M be a compact Kähler manifold. **Then any cohomology class can be represented as a sum of closed (p,q)-forms.**

dd^c-lemma

DEFINITION: Let M be a complex manifold, and $I: TM \longrightarrow TM$ its complex structure operator. **The twisted differential** of M is $IdI^{-1}: \Lambda^*(M) \longrightarrow \Lambda^{*+1}(M)$, where I acts on 1-forms as an operator dual to $I: TM \longrightarrow TM$, and on the rest of differential forms multiplicatively.

REMARK: Consider the Hodge decomposition of the de Rham differential, $d = \partial + \overline{\partial}$, where $\partial : \Lambda^{p,q}(M,I) \longrightarrow \Lambda^{p+1,q}(M,I)$ and $\overline{\partial} : \Lambda^{p,q}(M,I) \longrightarrow \Lambda^{p+1,q}(M,I)$. **Then $d = \operatorname{Re}\partial$ and $d^c = \operatorname{Im}\partial$.** Also, $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

THEOREM: (**dd^c-lemma**) Let η be a form on a compact Kähler manifold, satisfying one of the following conditions.
(1). η is an exact (p,q)-form. (2). η is d-exact, d^c-closed.
Then $\eta \in \operatorname{im} dd^c$.

Holomorphically symplectic manifolds

DEFINITION: Let (M, I) be a complex manifold, and $\Omega \in \Lambda^2(M, \mathbb{C})$ a differential form. We say that Ω is **non-degenerate** if $\ker \Omega \cap T_{\mathbb{R}} M = 0$. We say that it is **holomorphically symplectic** if it is non-degenerate, $d\Omega = 0$, and $\Omega(IX, Y) = \sqrt{-1}\,\Omega(X, Y)$.

REMARK: The equation $\Omega(IX, Y) = \sqrt{-1}\Omega(X, Y)$ means that **Ω is complex linear with respect to the complex structure on $T_{\mathbb{R}} M$ induced by I.**

REMARK: Consider the Hodge decomposition $T_{\mathbb{C}} M = T^{1,0} M \oplus T^{0,1} M$ (decomposition according to eigenvalues of I). Since $\Omega(IX, Y) = \sqrt{-1}\,\Omega(X, Y)$ and $I(Z) = -\sqrt{-1}\, Z$ for any $Z \in T^{0,1}(M)$, we have $\ker(\Omega) \supset T^{0,1}(M)$. Since $\ker \Omega \cap T_{\mathbb{R}} M = 0$, real dimension of its kernel is at most $\dim_{\mathbb{R}} M$, giving $\dim_{\mathbb{R}} \ker \Omega = \dim M$. **Therefore,** $\ker(\Omega) = T^{0,1} M$.

COROLLARY: Let Ω be a holomorphically symplectic form on a complex manifold (M, I). **Then I is determined by Ω uniquely.**

C-symplectic structures

DEFINITION: (Bogomolov, Deev, V.) Let M be a smooth $4n$-dimensional manifold. A complex-valued form Ω on M is called **almost C-symplectic** if $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is a non-degenerate volume form. It is called **C-symplectic** when it is also closed.

THEOREM: Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a C-symplectic form, and $T^{0,1}_\Omega(M)$ be equal to $\ker \Omega$, where $\ker\Omega := \{v \in TM \otimes \mathbb{C} \ \ | \ \ \Omega \lrcorner v = 0\}$. Then $T^{0,1}_\Omega(M) \oplus \overline{T^{0,1}_\Omega(M)} = TM \otimes_{\mathbb{R}} \mathbb{C}$, hence **the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable, and Ω is holomorphically symplectic on (M, I_Ω).**

Proof: Rank of Ω is $2n$ because $\Omega^{n+1} = 0$ and $\operatorname{Re}\Omega$ is non-degenerate. Then $\ker\Omega \oplus \overline{\ker\Omega} = T_{\mathbb{C}}M$. The relation $[T^{0,1}_\Omega(M), T^{0,1}_\Omega(M)] \subset T^{0,1}_\Omega(M)$ follows from Cartan's formula

$$d\Omega(X_1, X_2, X_3) = \frac{1}{6} \sum_{\sigma \in \Sigma_3} (-1)^{\tilde\sigma} \operatorname{Lie}_{X_{\sigma_1}} \Omega(X_{\sigma_2}, X_{\sigma_3}) + (-1)^{\tilde\sigma}\Omega([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3})$$

which gives, for all $X, Y \in T^{0,1}M$, and any $Z \in TM$,

$$d\Omega(X, Y, Z) = \Omega([X, Y], Z),$$

implying that $[X, Y] \in T^{0,1}M$. ■

Period map for holomorphically symplectic manifolds

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient CTeich $:= \frac{\text{CSymp}}{\text{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map CTeich $\longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

Injectivity of the period map immediately implied by the following version of Moser's lemma.

THEOREM: (Soldatenkov, V.)
Let (M, I_t, Ω_t), $t \in [0,1]$ be a family of C-symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \text{Diff}_0(M)$, such that $V_t^* \Omega_0 = \Omega_t$.**

Proof: Included in the slides below.

Local Torelli theorem

DEFINITION: Let M be a compact complex manifold. We say that M **satisfies $\partial\bar\partial$-lemma in term $\Lambda^{p,q}(M)$** if any ∂-closed, $\bar\partial$-exact (p,q)-form belongs to the image of $\partial\bar\partial$.

REMARK: As usual, we denote $H^q(\Omega^p M)$ by $H^{p,q}(M)$. Note that **this makes sense even when M does not admit the Hodge decomposition.**

One of our aims today is an explicit version of local Torelli theorem:

THEOREM: (Kurnosov, V.)
Let (M,Ω) be a C-symplectic manifold. Assume that $H^2(M)$ admits the Hodge decomposition, $H^{0,1}(M)=0$, $H^{2,0}(M)=\mathbb{C}$, and M satisfies the $\partial\bar\partial$-lemma in $\Lambda^{1,2}(M)$. Let $W:=\frac{H^2(M,\mathbb{C})}{\langle\overline\Omega\rangle}$. Then the period map composed with the natural projection $H^2(M,\mathbb{C})\mapsto W$ **defines a local difeomorphism from** CTeich **to W.**

Proof: Later today.

Holomorphically symplectic Moser's lemma

THEOREM: (Soldatenkov, V.)
Let (M, I_t, Ω_t), $t \in [0,1]$ be a family of C-symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \operatorname{Diff}_0(M)$, such that $V_t^*\Omega_0 = \Omega_t$.**

Proof. Step 1: If we find a vector field X_t such that $\operatorname{Lie}_{X_t}\Omega_t = \frac{d}{dt}\Omega_t$, we have (like in the proof of Moser's lemma)

$$V_{t_1}^*\Omega_0 = \int_0^{t_1} \operatorname{Lie}_{X_t}\Omega_t dt = \int_0^{t_1} \frac{d\Omega_t}{dt} dt = \Omega_{t_1}$$

where V_t is a diffeomorphism flow such that $\frac{dV_t}{dt} = X_t$. **It remains to find the family $X_t \in T_{\mathbb{R}}M$.**

Step 2: The contraction map $\Lambda^{2,0}M \otimes_{\mathbb{R}} T_{\mathbb{R}}M \longrightarrow \Lambda^{1,0}(M)$ **is surjective** (an exercise).

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t}\Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

Holomorphically symplectic Moser's lemma (2)

It remains to find $X_t \in T_{\mathbb R}M$ such that $\operatorname{Lie}_{X_t}\Omega_t = \frac{d}{dt}\Omega_t$.

Step 2: The contraction map $\Lambda^{2,0}M \otimes_{\mathbb R} T_{\mathbb R}M \longrightarrow \Lambda^{1,0}(M)$ **is surjective**.

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t}\Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

Step 4: Let $\Omega_t' := \frac{d}{dt}\Omega_t$ and $\dim_{\mathbb C} M = 2n$. Differentiating $\Omega_t^{n+1} = 0$ in t, we obtain $\Omega_t' \wedge \Omega_t^n = 0$. Since $\Phi := \Omega_t^n$ is a holomorphic volume form, the multiplication map $\Lambda^{0,2}(M) \xrightarrow{\wedge\Phi} \Lambda^{2n,2}(M)$ is an isomorphism of vector bundles. **Then $\Omega_t' \wedge \Omega_t^n = 0$ implies that $\Omega_t' \in \Lambda^{1,1}(M, I_{\Omega_t}) + \Lambda^{2,0}(M, I_{\Omega_t})$.**

Step 5: Using Step 3 and Step 4, we obtain that holomorphic Moser's lemma **is implied by the following statement.**

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

Holomorphically symplectic Moser's lemma (3)

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

Proof. Step 1: Let $\eta = d\beta$, where $\beta = \beta^{1,0} + \beta^{0,1}$. Since $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$, we have $\overline{\partial}(\beta^{0,1}) = 0$. The first cohomology of the complex $(\Lambda^{0,*}(M), \overline{\partial})$ vanish, because $H^{0,1}(M) = 0$, **hence $\beta^{0,1} = \overline{\partial}\psi$, for some $\psi \in C^\infty M$.**

Step 2: This gives $\eta = d(\beta - d\psi)$, hence $\alpha := \beta - d\psi = \beta^{1,0} + \beta^{0,1} - \partial\psi - \beta^{0,1}$ **is a (1,0)-form which satisfies $\eta = d\alpha$.** ■

COROLLARY: Let CSymp be the space of all C-symplectic structures with C^∞-topology. Denote by CTeich the corresponding Teichmüller space, $\text{CTeich} := \frac{\text{CSymp}}{\text{Diff}_0(M)}$. Define **the period map** $\text{Per}: \text{CTeich} \longrightarrow H^2(M, \mathbb{C})$ mapping Ω to its cohomology class. **Then Per is locally a homeomorphism to its image.**

Proof: All fibers of Per are 0-dimensional. ■

Local Torelli theorem for a K3 surface

REMARK: In real dimension 4, C-symplectic form **is a pair ω_1, ω_2 of symplectic forms which satisfy $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$.**

THEOREM: Let (M, I, Ω) be a complex holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Then for any sufficiently small cohomology class $[\eta] \in H^{1,1}(M)$, **there exists a C-symplectic form $\Omega + \rho$, where $\rho \in \Lambda^{1,1}M + \Lambda^{0,2}M$ is a closed form which satisfies $\rho^{1,1} \wedge \rho^{1,1} = -\Omega \wedge \rho^{0,2}$, and $\rho^{1,1}$ is ∂-cohomologous to $[\eta]$.** Moreover, the cohomology class of ρ **is uniquely determined by $[\eta]$**.

Proof: Next slide

REMARK: This theorem locally identifies $H^{1,1}(M)$ with the neighbourhood Ω in the C-symplectic Teichmüller space, proving that it is smooth and $b_2 - 2$-dimensional. **This proves the local Torelli theorem for K3.**

REMARK: The proof of this theorem is done using the same argument as used to prove the Maurer-Cartan equation, central to Kuranishi theory. Indeed, **the equation (*) we are going to solve below is a version of Maurer-Cartan, adopted and simplified for the C-symplectic structures.**

Local Torelli theorem for K3 (2)

THEOREM: Let (M, I, Ω) be a complex holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Then for any sufficiently small cohomology class $[\eta] \in H^{1,1}(M)$, **there exists a C-symplectic form $\Omega + \rho$, where $\rho \in \Lambda^{1,1}M + \Lambda^{0,2}M$ is a closed form which satisfies $\rho^{1,1} \wedge \rho^{1,1} = -\Omega \wedge \rho^{0,2}$, and $\rho^{1,1}$ is ∂-cohomologous to $[\eta]$.** Moreover, the cohomology class of ρ **is uniquely determined by $[\eta]$**.

Proof. Step 1: Since $(\Omega + \rho)^2 = \rho^{1,1} \wedge \rho^{1,1} + \Omega \wedge \rho^{0,2} = 0$, **this form is almost C-symplectic. To prove that it is C-symplectic, we need to find ρ such that $d\rho = 0$.**

Step 2: From Hodge to de Rham isomorphism, we obtain that the cohomology class $[u]$ of $\Omega + \rho$ is equal to $[\Omega + \eta + u^{0,2}]$. Since M is K3, we have $H^{0,2}(M) = \mathbb{C}[\overline{\Omega}]$, which gives $[u^{0,2}] = \lambda[\overline{\Omega}]$, for some $\lambda \in \mathbb{C}$. since $(\Omega + \rho)^2 = 0$, this gives $[\Omega \wedge u^{0,2}] = [\eta]$. Then $\lambda = -\frac{[\eta^2]}{[\Omega \wedge \overline{\Omega}]}$. **We proved that the cohomology class of $\Omega + \rho$ is uniquely determined by $[\eta^2]$.**

Local Torelli theorem for K3 (3)

Below, we need the following version of **$\partial\bar\partial$-lemma**: **for any $(1,2)$-form α, which is ∂-exact and $\bar\partial$-closed, $\alpha = \bar\partial\beta$, where β is ∂-exact.**

Step 3: Let Λ_Ω be contraction with the $(2,0)$-bivector associated with Ω. This operation clearly commutes with $\bar\partial$. Then $\rho^{1,1}\wedge\rho^{1,1} = -\Omega\wedge\rho^{0,2}$ is equivalent to $\Lambda_\Omega(\rho^{1,1}\wedge\rho^{1,1}) = -\rho^{0,2}$. **To solve the equation $d\rho = 0$, we solve the equivalent equation, which is a version of Maurer-Cartan**

$$\partial\Lambda_\Omega(\rho^{1,1}\wedge\rho^{1,1}) = -\bar\partial\rho^{1,1}, \qquad \partial\rho^{1,1} = 0. \qquad (*)$$

Let γ_0 be the harmonic (1,1)-form representing $[\eta]$. We solve the equation (*) inductively by taking

$$\bar\partial\gamma_n = \partial\Lambda_\Omega\left(\sum_{i+j=n-1}\gamma_i\wedge\gamma_j\right). \quad (**)$$

Such γ_n is found using $\partial\bar\partial$-lemma, because the RHS of (**) is ∂-exact and $\bar\partial$-closed, which is clear because $\bar\partial$ commutes with Λ_Ω. Since $\bar\partial\sum_i\gamma_i = \partial\Lambda_\Omega\left(\sum_{i,j}\gamma_i\wedge\gamma_j\right)$, the sum $\rho^{1,1} := \sum\gamma_i$ is a solution of (*).

Step 4: Since γ_i, $i > 0$ are ∂-exact, the ∂-cohomology class of γ is $[\gamma_0] = [\eta]$.

■

Almost C-symplectic forms

The proof of local Torelli for K3 involves solving an order 2 equation $\Omega\wedge\rho^{0,2} = -\rho^{1,1}\wedge\rho^{1,1}$, because the condition $\Omega^2=0$ is quadratic. To extend this argument to any dimension **we write a degree 2 polynomial equation which describes almost C-symplectic structures in the space of all complex-valued 2-forms.**

DEFINITION: Let V be a real vector space of dimension $4n$, and $\Lambda^2_{\mathbb C}V := \Lambda^2 V\otimes_{\mathbb R}\mathbb C$. A 2-form $\Omega\in\Lambda^2_{\mathbb C}V$ is **C-symplectic** if $\Omega^n\wedge\overline{\Omega}^n\neq 0$ and $\Omega^{n+1}=0$.

Claim 1: Fix a complex structure $I\in\operatorname{End}_{\mathbb R}V, I^2=-\operatorname{Id}$ on V, and let $\Theta\in\Lambda^2_{\mathbb C}V$ be a C-symplectic form. Denote the $(2,0)$-component of Θ by Ω, let $\eta^{1,1}$ be its $(1,1)$-component and $\eta^{0,2}$ be its $(0,2)$-component. Assume that Ω is C-symplectic, that is, has maximal rank. **Then**

$$\eta^{1,1}\wedge\eta^{1,1}\wedge\Omega^{n-1} = -\eta^{0,2}\wedge\Omega^n. \quad (*)$$

Moreover, **for any (1,1)-form $\eta^{1,1}$, there exists a unique $(0,2)$-form $\eta^{0,2}$ such that (*) holds.**

Proof. Step 1: The $(2n,2)$-component of Θ^{n+1} is equal to $\eta^{1,1}\wedge\eta^{1,1}\wedge\Omega^{n-2}+\eta^{0,2}\wedge\Omega^n$; now, $\Theta^{n+1}=0$ implies (*) immediately.

Step 2: The map $\Lambda^{0,2}V\xrightarrow{\wedge\Omega^n}\Lambda^{2n,2}V$ is clearly an isomorphism. Existence and uniqueness of $\eta^{0,2}$ solving (*) follows from this observation. ∎

The operator Λ_Ω

We obtained a quadratic equation (*) from an order $n+1$-equation $\Theta^{n+1} = 0$. **We are going to show that the converse is also true:** (*) implies $\Theta^{n+1} = 0$, at least in a neighbourxood of a C-symplectic structure $\Omega \in \Lambda^{2,0}_{\mathbb{C}} V$.

DEFINITION: Fix a complex structure $I \in \operatorname{End}_{\mathbb{R}} V, I^2 = -\operatorname{Id}$ on V, and let $\Omega \in \Lambda^{2,0}_{\mathbb{C}} V$ be a C-symplectic form. Consider a (2,2)-form $\Theta \in \Lambda^{2,2} V$ and let $u \in \Lambda^{0,2} V$ be a $(0,2)$-form which satisfies $u \wedge \Omega^n = \Theta \wedge \Omega^{n-1}$. By Step 2 of Claim 1, such u exists for any (2,2)-form Θ. **The map which takes Θ to u is denoted $\Theta \mapsto \Lambda_\Omega \Theta$.**

REMARK: Using this notation, **the equation (*) can be written as $\eta^{0,2} = -\Lambda_\Omega(\eta^{1,1} \wedge \eta^{1,1})$.**

Almost C-symplectic forms: reducing $\Theta^{n+1}=0$ to a quadratic equation

THEOREM A: Let V be a real vector space of dimension $4n$, and $I\in \operatorname{End}_{\mathbb R} V, I^2=-\operatorname{Id}$ a complex structure. Denote by Z the space of C-symplectic structures $\Theta\in\Lambda^2_{\mathbb C}V$, such that $\Theta^{2,0}$ is non-degenerate, and let Z_1 be the space of all triples $\Theta=\Omega+\eta^{1,1}+\eta^{0,2}$, where Ω is a non-degenerate (2,0)-form, and $\eta^{0,2}=-\Lambda_\Omega(\eta^{1,1}\wedge\eta^{1,1})$. **Then $Z=Z_1$ in a sufficiently small neighbourhood of a given non-degenerate (2,0)-form Θ_0.**

Proof. Step 1: By Claim 1, any $\Theta\in Z$ written as $\Theta=\Omega+\eta^{1,1}+\eta^{0,2}$ satisfies $\eta^{0,2}=-\Lambda_\Omega(\eta^{1,1}\wedge\eta^{1,1})$, hence **the set Z_1' of all $\Theta\in Z$ such that $\Theta^{2,0}$ is non-degenerate lies in Z_1.** However, clearly, $Z_1'\subset Z$ is open, hence **locally around a given $\Omega\in\Lambda^{2,0}(V)$, we have $Z\subset Z_1$.**

Step 2: The space of pairs $(\Omega,\eta^{1,1})\subset\Lambda^{2,0}(V)\oplus\Lambda^{1,1}(V)$, where $\Omega\in\Lambda^{2,0}(V)$ is a non-degenerate form, is open, hence $Z_1\subset\Lambda^2_{\mathbb C}(V)$ is a smooth submanifold in $\Lambda^2_{\mathbb C}(V)$. On the other hand, Z is homogeneous under the $GL_{\mathbb R}(V)$-action, hence Z is also smooth. **To show that $Z=Z_1$, it would suffice to prove that $\dim Z=\dim Z_1$.**

Step 3: The real dimension of Z is $\dim_{\mathbb R} GL_{\mathbb R}(V)-\dim_{\mathbb R} Sp_{\mathbb C}(V,\Omega)=16n^2-2\frac{(2n+1)2n}{2}=16n^2-4n^2-2n=12n^2-2n$. The real dimension of Z_1 is $\dim_{\mathbb R}\Lambda^{2,0}_{\mathbb C}V+\dim_{\mathbb R}\Lambda^{1,1}_{\mathbb C}V=2\frac{(2n-1)2n}{2}+2(2n)^2=4n^2-2n+8n^2=12n^2-2n$. This gives $\dim Z=\dim Z_1$ and (locally) $Z=Z_1$ (Step 2). ■

The local Torelli theorem for C-symplectic manifolds

THEOREM: ("Local Torelli theorem"; Kurnosov, V.)
Let (M, Ω) be a C-symplectic manifold. Assume that $H^{0,1}(M) = 0$, $H^{2,0}(M) = \mathbb{C}$ and M satisfies $\partial\bar\partial$-lemma in $\Lambda^{1,2}(M)$ and has Hodge decomposition in $H^2(M)$. Let $W := \frac{H^2(M,\mathbb{C})}{\langle\overline{\Omega}\rangle}$. Then the period map composed with the natural projection $H^2(M, \mathbb{C}) \mapsto W$ **defines a local difeomorphism from** CTeich **to a neighbourhood of 0 in W.**

Proof. Step 1: Local injectivity of the period map **follows immediately if we prove that the space** CSymp **of C-symplectic structures is smooth and the map** Per : CSymp $\longrightarrow H^2(M, \mathbb{C})$ **is submersive onto its image, which is also smooth.** Indeed, in this case the fibers of Per are locally connected, and C-symplectic Moser lemma implies that Diff_0-action on $\operatorname{Per}^{-1}([\Omega])$ is locally transitive.

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Step 2: The image of $\operatorname{Per}: \operatorname{CSymp} \longrightarrow H^2(M,\mathbb{C})$ is the space of all cohomology classes $\underline{\Theta} = \lambda\underline{\Omega} + \underline{\eta}^{1,1} + \underline{\eta}^{0,2}$ such that

$$(\lambda\underline{\Omega})^n \wedge \underline{\eta}^{2,0} = -(\lambda\underline{\Omega})^n \wedge \underline{\eta}^{1,1} \wedge \underline{\eta}^{1,1}.$$

This relation is obtained as $(2n,2)$-part of $(\underline{\Theta})^{n+1} = 0$. For any $\underline{\Theta}$ close to $\underline{\Omega}$ which satisfies this relation **we obtain a solution of the C-symplectic Maurer-Cartan equation (*) with the same cohomology class, using the same argument as for K3 surface**. We use Theorem A to infer that this solution is C-symplectic. This argument proves that the natural projection $H^2(M,\mathbb{C}) \mapsto W$ is locally surjective, and, indeed, submersive. Using the C-symplectic Moser lemma as in Step 1, we also obtain that the period map $\operatorname{CTeich} \longrightarrow H^2(M,\mathbb{C})$ is locally injective. ■

REMARK: This theorem can be applied to all compact hyperkähler manifolds of maximal holonomy, and to the **Bogomolov-Guan manifolds,** which are simply connected, holomorphically symplectic, compact and non-Kähler.

The Bogomolov's local Torelli theorem

Consider a holomorphically symplectic manifold (M, Ω) with $\dim H^{2,0}(M) = 1$. Consider the Teichmüller space Teich of its complex deformations which are holomorphically symplectic and also satisfy $\dim H^{2,0}(M) = 1$. Note that CTeich **is projected to** Teich **with fibers of complex dimension 1,** because (M, Ω) and $(M, \lambda\Omega) \in$ CTeich correspond to the same point of Teich.

Definition: Let $\operatorname{Per}: \operatorname{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map a holomorphically symplectic structure J to a point $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega\rangle \in \operatorname{Gr}(2, H^2(M, \mathbb{R}))$ The map Per_B : $\operatorname{Teich} \longrightarrow \operatorname{Gr}(2, H^2(M, \mathbb{R}))$ is called **the Bogomolov period map**.

THEOREM: (local Torelli theorem for Bogomolov period map) Let M be a holomorphically symplectic manifold which satisfies the assumptions of the local Torelli theorem. Then the Bogomolov period map Per_B : $\operatorname{Teich} \longrightarrow \operatorname{Gr}(2, H^2(M, \mathbb{R}))$ **is locally a diffeomorphism.**

Proof: Since Teich is smooth, it would suffice to show that Per_B induces an isomorphism of tangent spaces. Let $W \subset H^2(M, \mathbb{R})$ be the subspace generated by $\operatorname{Re}\Omega, \operatorname{Im}\Omega$. Then $T_W \operatorname{Gr}(2, H^2(M, \mathbb{R}))$ is $\operatorname{Hom}(W, H^2(M, \mathbb{R})/W)$. By local Torelli theorem, the tangent space to Teich is identified with $\frac{H^2(M,\mathbb{C})}{\langle \Omega, \overline{\Omega}\rangle}$. This is the same as $(H^2(M, \mathbb{R})/W) \otimes_{\mathbb{R}} \mathbb{C}$, and the same as $\operatorname{Hom}(W, H^2(M, \mathbb{R})/W) = T_{\underline{\Omega}} \operatorname{Gr}(2, H^2(M, \mathbb{R}))$. ■

Polynomial invariants of Lie groups

PROPOSITION 1: Let V be a real vector space equipped with an action of a Lie group G, and Q a G-invariant polynomial function. Let $S\subset \operatorname{Gr}(2,V)$ be an open subset in the Grassmannian of 2-planes. Assume that for any $W\in S$, there exists a subgroup $\rho_W\subset G$ isomorphic to S^1 acting by rotations on W and trivially on V/W. **Then Q is proportional to q^n, where q is a quadratic form on V.**

Proof. Step 1: Let $W\in S$ be a 2-plane in V. **Any rotation-invariant polynomial function on $\mathbb{R}^2$ is a power of quadratic form (prove this as an exercise)**, hence $Q|_W=\lambda q_W^n|_W$, for some quadratic form q_W.

Step 2: We want to take the n-th root of Q. When n is odd, the n-th root of Q is well defined. When n is even, the restriction $Q|_W$ does not change sign, hence Q does not change sign on the set $U_S\subset V$ of all vectors passing through planes $W\in S$. **The function $q:=\sqrt[n]{\pm Q}$ is well defined on the whole of V when n is odd, and on an open subset $U_S\subset V$ when it is even.**

Polynomial invariants of Lie groups (2)

PROPOSITION 1: Let V be a real vector space equipped with an action of a Lie group G, and Q a G-invariant polynomial function. Let $S \subset \mathrm{Gr}(2, V)$ be an open subset in the Grassmannian of 2-planes. Assume that for any $W \in S$, there exists a subgroup $\rho_W \subset G$ isomorphic to S^1 acting by rotations on W and trivially on V/W. **Then Q is proportional to q^n, where q is a quadratic form on V.**

Step 2: The function $q := \sqrt[n]{\pm Q}$ is well defined on the whole of V when n is odd, and on an open subset $U_S \subset V$ when it is even.

Step 3: The function $q: U_S \longrightarrow \mathbb{R}$ is a polynomial of second degree on all hyperplanes $W \in S$. Consider the second derivative $H := \frac{d^2}{dxdy} q$ as a section of $\mathrm{Sym}^2 T^* U_S$. Take $\zeta \in T_v U_S = V$ such that $\langle \zeta, v \rangle \in S$. Since q is a quadratic function on $\langle \zeta, v \rangle$, the value of the function $v \mapsto H(\zeta, \zeta)$ is independent from v. The set of ζ for which this is true is open, and $H(\zeta, \zeta)$ is a quadratic polynomial on ζ. **This implies that $v \longrightarrow H(\zeta, \zeta)$ is constant on U_S, for any $\zeta \in V$.**

Step 4: A function which satisfies $\frac{d^2}{dxdy} q = const$ is a quadratic polynomial. We extend it to a quadratic polynomial on V. **Then $Q = \lambda q^n$ on U_S.** Since Q is polynomial, and $U_S \subset V$ is open, this expression is true everywhere. ■

The BBF form

THEOREM: Let M be a compact holomorphically symplectic manifold, $\dim_{\mathbb{C}} M = 2n$, admitting the Hodge decomposition on $H^2(M)$. Assume that M satisfies the $\partial\bar{\partial}$-lemma in $H^{1,2}(M)$ and $H^{0,2}(M) = H^{2,0}(M) = \mathbb{C}$. **Then the space $H^2(M)$ is equipped with a bilinear symmetric form q which vanishes on the image of** Per $CTeich \longrightarrow H^2(M, \mathbb{C})$**.** Moreover, **for any $\eta \in H^2(M)$, one has $\int_M \eta^{2n} = q(\eta, \eta)^n$.**

Proof. Step 1: Consider the Hodge decomposition on $H^2(M)$ induced by the complex structure $I \in U$. This gives **"the Hodge rotation map"**, that is, an $U(1)$-action $\rho_I(t)$, acting as $e^{2\pi\sqrt{-1}\,(p-q)t}$ on $H^{p,q}(M)$. **Clearly, the polynomial $Q(\eta) := \int_M \eta^{2n}$ is ρ_I-invariant.** By definition, ρ_I acts trivially on $H^{1,1}(M)$ and rotates $W = \langle \operatorname{Re}\Omega, \operatorname{Im}\Omega\rangle$.

Step 2: Let G be the Lie group generated by the Hodge rotation maps ρ_I for all complex structures I satisfying the assumptions of the theorem. Since the image of the period map is open, **the action of G satisfies assumptions of Proposition 1, giving $Q(\eta) = \lambda q(\eta, \eta)^n$.** ■

DEFINITION: Usually one normalizes q in such a way that it is integer and primitive; then $Q(\eta) = \lambda q(\eta, \eta)^n$, where $\lambda > 0$ is called **the Fujiki constant**. The form q is called **the Bogomolov-Beauville-Fujiki form** (the BBF form).

Holomorphic Lagrangian subvarieties

DEFINITION: Let (M,Ω) be a holomorphically symplectic manifold, and $X\subset (M,\Omega)$ a complex subvariety. It is called **holomorphic Lagrangian** if Ω restricted to the set of smooth points of X vanishes.

PROPOSITION: (Hitchin's lemma)
Let $X\subset M$ be a real submanifold (or closed real analytic subvariety) such that $\Omega|_X=0$ and $\dim_{\mathbb R} X=\frac{1}{2}\dim_{\mathbb R} M$. **Then X is a complex subvariety.**

Proof. Step 1: This statement would follow if we prove the following linear-algebraic statement. Let (V,Ω) be a real vector space equipped with a C-symplectic form, $I: \ V\longrightarrow V$ the induced complex structure operator, and $W\subset V$ a real subspace such that $\dim_{\mathbb R} W=\frac{1}{2}\dim_{\mathbb R} V$ and $\Omega|_W=0$. **Then $I(W)=W$**, that is, **W is a complex subspace of V.**

Proof: Let $u,w\in W$. Since Ω is I-linear, one has $0=\sqrt{-1}\Omega(u,w)=\Omega(Iu,w)$, hence the space $W_u:=\langle W+Iu\rangle$ generated by W and Iu is Lagrangian with respect to $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$. Since the forms $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$ are non-degenerate, dimension of W_u cannot be bigger than $\dim_{\mathbb R} W=\frac{1}{2}\dim_{\mathbb R} V$, hence $W_u=W$.
■

"Good" subvarieties

DEFINITION: Consider a closed complex subvariety $X \subset M$, and let $\tilde M$ a blow-up of M such that the proper preimage of X is a subvariety $\tilde X \subset \tilde M$ which has simple normal crossings. The **essential skeleton** of X is a CW-complex associated with $\tilde X$ as follows: its vertices are irreducible components of $\tilde X$, and its k-simplexes with vertices associated to the components $X_1, ..., X_k$ are irreducible components of the intersection $\bigcup_{i=1}^k X_k$. By a theorem of D. A. Stepanov, **the homotopy type of the essential skeleton is independent from the choice of resolution.** We call a function f on X **smooth** if its pullback to $\tilde X$ is smooth.

DEFINITION: A closed, compact subvariety $X \subset M$ is called **good** if the resolution of X is Kähler, any smooth function on X can be extended to a smooth function in a neighbourhood of X in M, and the essential skeleton S of $\tilde X$ satisfies $H^1(S) = 0$.

dd^c-lemma and essential skeleton

Proposition 4: Let η be a (1,1)-form on M which is exact on $X \subset M$, which is a "good" complex subvariety. **Then $\eta = dd^c f$ in a neighbourhood of X.**

Proof. Step 1: Let $\tilde X_0$ be the resolution of X, obtained from $\tilde X$ by taking apart the branches. **Then the pullback η_0 of η to $\tilde X_0$ is dd^c-exact,** because $\tilde X_0$ is smooth and Kähler, hence $\eta_0 = dd^c f$.

Step 2: Let $\pi : \tilde X_0 \longrightarrow X$ be the projection, and $x \in X$ any point. Since $\ker dd^c$ is holomorphic plus antiholomorphic functions, on a compact complex variety $\ker dd^c$ is constant functions. On each irreducible component of $\pi^{-1}(x)$, the function f satisfies $dd^c f = 0$, hence it is constant. Therefore, the function f such that $\eta_0 = dd^c f$ is uniquely, up to a constant, defined on each connected component X_i of $\tilde X_0$. To show that f is a pullback of a function on $\tilde X$, and hence on X, **we need to chose these constants in such a way that $f|_{X_i}$ agrees on all intersections $X_i \cap X_j$.**

Step 3: Choose a function f_i which satisfies $dd^c f_i = \eta_0|_{X_i}$ on each of these components. The difference $f_i|_{X_i \cap X_j} - f_j|_{X_i \cap X_j}$ is a constant function on each intersection $X_i \cap X_j$ which sums up to zero on triple intersections, hence it defines a 1-cocycle on S. **To choose f_i which agree on intersections, we need to show that this cocycle is exact;** this can be ensured by assuming that $H^1(S) = 0$, where S is the essential skeleton. ■

Deformations of Lagrangian subvarieties

THEOREM: (joint with N. Kurnosov)
Let (M, Ω) be a compact C-symplectic manifold satisfying the assumptions of local Torelli theorem, $X \subset (M, \Omega)$ a good closed holomorphic Lagrangian subvariety. Consider the space $\mathrm{CTeich}_X \subset \mathrm{CTeich}$ consisting of all $\Omega' \in \mathrm{CTeich}$ such that the restriction of Ω' to X is exact. Assume that X is "good" in the sense of the above definition. Then locally around $\Omega \in \mathrm{CTeich}_X$ **there exist a choice of holomorphic symplectic representatives Ω_t, smoothly depending on $t \in \mathrm{CTeich}_X$, such that $\Omega_t|_X = 0$ for all t.**

Proof: Next slide.

REMARK: In other word, for a sufficiently small deformation $\Omega_t \in \mathrm{CTeich}_X$ of the C-symplectic structure Ω in CTeich_X, **the variety X can be deformed to a Lagrangian subvariety in (M, Ω_t).**

REMARK: This result was proven by Voisin for smooth holomorphic Lagrangian X in projective M, and by C. Lehn when X are SNC holomorphic Lagrangian subvarieties in projective M. **We needed this result for Bogomolov-Guan manifolds**, and found **an improved proof of Voisin's theorem which also works for singular X and non-Kähler M.**

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Proof. Step 1: After rescaling, we may assume that $[\Omega_t]^{2,0} = \Omega$. We write Ω_t by solving (*) recursively, starting with a closed $(1,1)$-form γ_0 representing $[\Omega_t]^{1,1}$:

$$\bar\partial\gamma_n = \partial\Lambda_\Omega\left(\sum_{i+j=n-1}\gamma_i\wedge\gamma_j\right). \quad (**)$$

Adding $dd^c f_n$ to γ_n won't affect (**). I will show that **we can always choose γ_n such that $\gamma_n|_X = 0$ if $\gamma_i|_X = 0$ for all $i < n$.**

Deformations of Lagrangian subvarieties (2)

Step 2: We use induction in n. Since X is "good", it satisfies the dd^c-lemma. By Proposition 4, we can replace γ_0 by $\gamma_0 - dd^c f$ such that $dd^c f|_X = \gamma_0|_X$. Smoothly extending f to a neighbourhood of X and replacing γ_0 by $\gamma_0 - dd^c f$, we obtain another closed representative of $[\gamma_0]$ which satisfies $\gamma_0|_X = 0$. **This is the basis of induction.**

Step 3: We need the following linear-algebraic lemma. Let $X \subset (M, \Omega)$ be a holomorphic Lagrangian subvariety, and $\alpha_1, \alpha_2 \in \Lambda^{1,1}(M)$ (1,1)-forms which satisfty $\alpha_1|_X = \alpha_2|_X = 0$. **Then $\Lambda_\Omega(\alpha_1 \wedge \alpha_2)|_X = 0$.** This is a local statement; using Darboux theorem, we introduce the coordinates such that $\Omega = \sum_i dp_i \wedge dq_i$ and all q_i are constant on X. Then $\alpha_k = \sum_{i,j} a_{ijk} dp_i \wedge d\overline{q}_j + \sum_{i,j} b_{ijk} dq_i \wedge d\overline{p}_j + \sum_{i,j} c_{ijk} dq_i \wedge d\overline{q}_j$, which gives

$$\Lambda_\Omega(\alpha_1 \wedge \alpha_2) = -\sum_{i,j,j'} a_{ijk} b_{ij'k} d\overline{q}_j \wedge d\overline{p}_{j'} - \sum_{i,j,j'} a_{ijk} c_{ij'k} d\overline{q}_j \wedge d\overline{q}_{j'}.$$

This form vanishes on X because $\overline{q}_j$ is constant on X.

Step 4: Suppose that $\gamma_i|_X = 0$ for all $i < n$. Then $u := \Lambda_\Omega \left(\sum_{i+j=n-1} \gamma_i \wedge \gamma_j\right)|_X = 0$ (Step 3). Solving the equation $\overline{\partial}\gamma_n = \partial u$, $\gamma_n \in \operatorname{im} \partial$, we obtain a solution γ_n which is ∂-exact and $\overline{\partial}$-closed on X. Applying Proposition 4 again, **we replace γ_n by another solution $\gamma_n - dd^c f$ which vanishes on X.** ■