Extremal metrics in quaternionic geometry

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Hypercomplex manifolds

DEFINITION: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$. Suppose that I, J, K are integrable almost complex structures. Then (M, I, J, K) is called **a hypercomplex manifold**.

THEOREM: (Obata, 1955) On any hypercomplex manifold **there exists a unique torsion-free connection ∇ such that $\nabla I = \nabla J = \nabla K$.**

DEFINITION: Such a connection is called **the Obata connection**.

REMARK: The holonomy of Obata connection lies in $GL(n, \mathbb{H})$.

REMARK: A torsion-free connection ∇ on M with $\mathcal{H}\mathrm{ol}(\nabla) \subset GL(n, \mathbb{H})$ **defines a hypercomplex structure on M.**

Examples of hypercomplex manifolds

EXAMPLE: A **Hopf surface** $M = \mathbb{H}\backslash 0/\mathbb{Z} \cong S^1 \times S^3$. The holonomy of Obata connection $\mathcal{H}\mathrm{ol}(M) = 0$.

EXAMPLE: **Compact holomorphically symplectic manifolds are hyperkähler (by Calabi-Yau theorem)**, hence hypercomplex. Here $\mathcal{H}\mathrm{ol}(M) \subset Sp(n)$ **(this is equivalent to being hyperkähler).**

PROPOSITION: A compact hypercomplex manifold (M, I, J, K) with (M, I) of Kähler type also admits a hyperkähler structure.

REMARK: In dimension 1, compact hypercomplex manifolds are classified (C. P. Boyer, 1988). This is the complete list: **torus, K3 surface, Hopf surface**.

Examples of hypercomplex manifolds (2)

EXAMPLE: The Lie groups

$$\begin{array}{lll} SU(2l+1), & T^1 \times SU(2l), & T^l \times SO(2l+1), \\ T^{2l} \times SO(4l), & T^l \times Sp(l), & T^2 \times E_6, \\ T^7 \times E^7, \quad T^8 \times E^8, & T^4 \times F_4, & T^2 \times G_2. \end{array}$$

Some other homogeneous spaces (D. Joyce and physicists Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen). **Holonomy unknown (but likely $GL(n,\mathbb{H})$).**

EXAMPLE: Many **nilmanifolds** (quotients of a nilpotent Lie group by a cocompact lattice) admit hypercomplex structures. **In this case $\mathcal{H}ol(M) \subset SL(n,\mathbb{H})$.**

Quaternionic Hermitian structures

DEFINITION: Let (M, I, J, K) be a hypercomplex manifold, and g a Riemannian metric. We say that g is **quaternionic Hermitian** if I, J, K are orthogonal with respect to g.

CLAIM: Quaternionic Hermitian metrics always exist.

Proof: Take any Riemannian metric g and **consider its average** $\mathrm{Av}_{SU(2)}\, g$ with respect to $SU(2) \subset \mathbb{H}^*$. ■

Given a quaternionic Hermitian metric g on (M, I, J, K), consider its Hermitian forms

$$\omega_I(\cdot,\cdot) = g(\cdot, I\cdot), \omega_J, \omega_K$$

(real, but *not closed*). Then $\Omega = \omega_J + \sqrt{-1}\omega_K$ is of Hodge type (2,0) with respect to I.

If $d\Omega = 0$, (M, I, J, K, g) is hyperkähler (this is one of the definitions).

Consider a weaker condition:

$$\partial\Omega = 0, \quad \partial : \Lambda^{2,0}(M, I) \longrightarrow \Lambda^{3,0}(M, I)$$

HKT structures

DEFINITION: (Howe, Papadopoulos, 1998)
Let (M, I, J, K) be a hypercomplex manifold, g a quaternionic Hermitian metric, and $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ the corresponding $(2,0)$-form. We say that g is **HKT ("hyperkähler with torsion")** if $\partial\Omega = 0$..

HKT-metrics play in hypercomplex geometry the same role as Kähler metrics play in complex geometry.

1. **They admit a smooth potential** (locally). There is a notion of an "HKT-class" (similar to Kähler class) in a certain finite-dimensional cohology group. Two metrics in the same HKT-class differ by a potential, which is a function.

2. When (M, I) has trivial canonical bundle, **a version of Hodge theory is established** giving an $\mathfrak{sl}(2)$-action on holomorphic cohomology $H^*(M, \mathcal{O}_{(M,I)})$.

HKT potential

Defining Kähler metric via Kähler potentials: A Kähler metric on (M,I) is one which is locally given as

$$g(\cdot,\cdot) = \sqrt{-1}\,\partial\overline{\partial}\varphi(\cdot, I\cdot)$$

where φ is a function called **a Kähler potential**.

Defining HKT metric through HKT potentials: An HKT metric on (M,I) is one which is locally given as

$$g(\cdot,\cdot) = D(\varphi), \quad \text{where} \quad D(\varphi) := \operatorname{Av}_{SU(2)}(\sqrt{-1}\,\partial\overline{\partial}\varphi(\cdot, I\cdot))$$

and φ is a function called **an HKT potential**.

THEOREM: (Banos-Swann)
This definition is equivalent to the usual one.

DEFINITION: A function which is an HKT potential of some HKT metric is called **strictly $\mathbb{H}$-plurisubharmonic**, or $\mathbb{H}$-psh.

REMARK: For a $\mathbb{H}$-psh function φ, **at least 1/4 eigenvalues of** Hess(φ) **must be positive.** Therefore, **there are no globally defined $\mathbb{H}$-psh functions** on compact manifolds.

HKT forms

DEFINITION: Let g be an HKT metric. The corresponding $(2,0)$-form $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ is called **an HKT form**.

CLAIM: Consider the multiplicative action of J on $\Lambda^*(M)$. **Then J maps $\Lambda^{p,q}(M))$ to $\Lambda^{q,p}(M))$.**

Proof: I and J anticommute. ■

DEFINITION: A (2,0)-form Ω on (M,I) is called **real** if $J(\Omega) = \overline{\Omega}$ and **positive** if $\Omega(x, J(\overline{x})) > 0$ for each non-zero $x \in T_I^{1,0}(M)$.

CLAIM: **Any HKT form is positive and real.** Moreover, **any ∂-closed positive real form $\Omega \in \Lambda_I^{2,0}(M)$ defines an HKT-metric** $g(x,y) := \Omega(x, J(\overline{y}))$.

An HKT cone.

Let g, g' be HKT metrics. We say that they are equivalent if $g = g' + D(\varphi)$ for some globally defined potential.

DEFINITION: An HKT cone is the set of all HKT metrics up to this equivalence.

CLAIM: Let g, g' be HKT metrics, with $g = g' + D(\varphi)$. **Then the corresponding HKT forms are related as $\Omega = \Omega' + \partial\partial_J\varphi$**, where $\partial_J(\varphi) := J(\overline{\partial}\varphi)$.

COROLLARY: An HKT cone is an open, convex subset in the cohomology group

$$\mathcal{H}(M) := \frac{\Lambda^{2,0}(M,\mathbb{R})_{\partial-\text{closed}}}{\partial\partial_J(C^\infty M)}.$$

This complex is elliptic, hence **$\mathcal{H}(M)$ is finite-dimensional when M is compact.**

MAIN QUESTION: Given a class $[\Omega]$ in the HKT cone, **find a privileged (extremal) metric in this class.**

Canonical bundle of a hypercomplex manifold.

0. Quaternionic Hermitian structure always exists.

1. **Complex dimension is even.**

2. **The canonical line bundle $\Lambda^{n,0}(M,I)$ of (M,I) is always trivial topologically.** Indeed, a non-degenerate section of canonical line bundle is provided by top power of a form Ω associated with some quaternionic Hermitian strucure. In particular, $c_1(M,I) = 0$.

3. Canonical bundle **is non-trivial holomorphically** in many cases. However, **when M is a nilmanifold**, $\Lambda^{n,0}(M,I)$ is trivial, and holonomy of Obata connection lies in $SL(n,\mathbb{H})$ (Barberis-Dotti-V., 2007)

4. If $\mathcal{H}\mathrm{ol}(M)$ lies in $SL(n,\mathbb{H})$, canonical bundle is trivial. The converse is true when M is compact and HKT (V., 2004): **an HKT manifold with holomorphically trivial canonical bunlde satisfies $\mathcal{H}\mathrm{ol}(M) \subset SL(n,\mathbb{H})$.**

HKT manifolds with trivial canonical bundle.

THEOREM: Let (M, I, J, K, Ω) be an HKT-manifold, $\dim_{\mathbb{H}} M = n$. Then **the following conditions are equivalent.**

1. $\overline{\partial}(\Omega^n) = 0$: this means that Ω^n **is a holomorphic section of a canonical bundle on** (M, I)

2. $\nabla(\Omega^n) = 0$, where ∇ is the Obata connection. **This implies, in particular, that** $\mathcal{H}ol(\nabla) \subset SL(n, \mathbb{H})$**.**

3. The manifold (M, I) with the induced quaternionic Hermitian metric is **balanced** (in the sense of Hermitial geometry): $d(\omega_I^{2n-1}) = 0$.

DEFINITION: An HKT metric satisfying any of these conditions is called **a Calabi-Yau HKT metric**.

REMARK: It is obtained as a solution of the **quaternionic Monge-Ampere equation**. In particular, **such a metric is unique in its cohomology class** (existence is conjectured).

HKT-Einstein manifolds

REMARK: Solving the quaternionic Monge-Ampere equation **gives an extremal metric for HKT manifolds with trivial canonical bundle** (analogue of Calabi-Yau manifolds). For non-trivial canonical bundle, the problem is more delicate.

REMARK: Let $\eta \in \Lambda^{1,1}(M,I)$ be a (1,1)-form, associated with a metric g. Then $J(\eta)$ is also a (1,1)-form, and it is positive if η is positive. The Hermitian form of $g' := \mathrm{Av}_{SU(2)}(g)$ is written as $\eta' := \eta + J(\eta)$.

DEFINITION: A real form $\eta \in \Lambda^{1,1}(M,I)$ is called **$\mathbb{H}$-positive** if $\eta + J(\eta)$ is a positive (1,1)-form.

DEFINITION: Let (M,I,J,K,g) be an HKT manifold, $\Omega^n(M,I)$ its canonical bundle with induced metric, and ρ its curvature. Then M is called **HKT-Einstein** if $\rho + J(\rho) = \lambda\omega_I$, where ω_I is the Hermitian form of (M,I), and $\lambda \in \mathbb{R}$.

REMARK: When $\Omega^n(M,I)$ admits a metric with $\mathbb{H}$-positive curvature, **uniqueness of HKT-Einstein metrics is easy to check**, existence is conjectured. When the curvature is $\mathbb{H}$-negative, the problem is similar to Fano case (quite hard).

Quaternionic Monge-Ampere equation

Let M be an HKT-manifold with holonomy in $SL(n,\mathbb{H})$ **(this is equivalent to having trivial canonical bundle).** Then the canonical bundle is trivialized by a form $\Phi_I \in \Lambda^{2n,0}$, non-degenerate, closed and satisfying $J(\Phi_I) = \overline{\Phi}_I$.

Quaternionic Monge-Ampere equation:

$$(\Omega + \partial\partial_J\varphi)^n = A_f e^f \Phi_I \quad (*)$$

where $\Omega+\partial\partial_J\varphi$ is an HKT-form. Here φ is unknown, and A_f is a number determined from

$$\int_M \Omega^n \wedge \overline{\Phi}_I = A_f \int_M e^f \Phi_I \wedge \overline{\Phi}_I$$

Theorem: (Alesker, V.) The solution φ of (*) is unique, if exists. Moreover, any solution of (*) admits a C^0-estimation in terms of f, Φ_I, Ω.

Conjecture: ("hypercomplex Calabi-Yau")
The equation (*) has a solution for all f, Φ_I, Ω.

Uniqueness of solutions of Monge-Ampere equations

Suppose Ω_1, Ω_2 are HKT-forms which are solutions of M-A, $\Omega_1 - \Omega_2 = \partial\partial_J \varphi$. Then $\Omega_1^n - \Omega_2^n = 0$. This gives

$$0 = \Omega_1^n - \Omega_2^n = \partial\partial_J \varphi \wedge \sum_{i=0}^{n-1} \Omega_1^i \wedge \Omega_2^{n-1-i}.$$

Denote by P the form $\sum_{i=0}^{n-1} \Omega_1^i \wedge \Omega_2^{n-1-i}$ and consider the differential operator $D: C^\infty(M) \longrightarrow C^\infty(M)$

$$\varphi \longrightarrow \frac{\partial\partial_J \varphi \wedge P}{\Omega^n}.$$

Then D is a second order operator with positive symbol.

Solutions of $D(f) = 0$ cannot have local maxima ("generalized maximum principle"). Since M is compact, **all solutions of $D(f) = 0$ are constant.**

A Lagrangian calibration form and quaternionim Monge-Ampere

The group $SU(2)$ of unitary quaternions acts on TM. By multilinearity, this action is extended to $\Lambda^*(M)$.

THEOREM: Let (M, I, J, K, g) be an $SL(n, \mathbb{H})$-manifold and $\tilde{\Psi} \in \Lambda^{2n}(M)$ a $2n$-form which is the real part of the holomorphic section of the canonical bundle on (M, J). Denote by Ψ the (p,p)-part of $\tilde{\Psi}$ with respect to I. **Then Ψ is a positive, closed form,** and for any $\varphi \in C^\infty M$ one has

$$(\Omega + \partial\partial_J\varphi)^n \wedge \overline{\Omega}^n = C(\omega_I + dd^c\varphi)^n \wedge \Psi,$$

where C is a positive constant.

COROLLARY: The quaternionic Monge-Ampere equation **is equivalent to a generalized Hessian equation** of form

$$(\omega_I + dd^c\varphi)^n \wedge \Psi = A_f e^f \operatorname{Vol}_M$$

with this particular Ψ.