Calabi-Yau theorem on Vaisman manifolds

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The Kähler potential and the Hessian

DEFINITION: A $(1,1)$-form ω on an almost complex manifold M is called **a Hermitian form** if $\omega(Ix, x) > 0$ for each non-zero tangent vector $x \in T_m M$. The corresponding Riemannian form $d(x, x) := \omega(Ix, x)$ is called **a Hermitian metric**.

DEFINITION: Let (M, I) be a complex manifold, and $d^c := IdI^{-1}$. The map $f \rightarrow dd^c f$ taking a function f to $dd^c f \in \Lambda^{1,1}(M)$ is called **the pluri-Laplacian**. A Hermitian form which can be locally represented as $dd^c f$ is called **Kähler** and f its **Kähler potential**. In this situation f is called **strictly plurisubharmonic**.

REMARK: Plurisubharmonicity is a weaker form of convexity: **every convex function is plurisubharmonic.**

DEFINITION: A **Kähler class** of a Kähler manifold is the cohomology class of its Kähler form.

The complex Monge-Ampère equation

QUESTION: What parametrizes Kähler forms in the same Kähler class?

ANSWER: A Kähler form is uniquely determined by its cohomology class and its volume form.

THEOREM: (dd^c-lemma)
Let ω_1, ω_2 be two cohomologous (1,1)-forms on a compact complex manifold. **Then $\omega_1 - \omega_2 = dd^c f$ for some function $f \in C^\infty M$.**

DEFINITION: The equation $(\omega + dd^c\varphi)^n = Ae^f\omega^n$, where A is a constant and $f \in C^\infty M$ is a given function is called **the Monge-Ampere equation,** and (φ, A) its solution. Note that $\omega + dd^c\varphi$ **is always a Kähler form,** otherwise $(\omega + dd^c\varphi)^n$ is degenerate somewhere on M.

THEOREM: (Calabi-Yau)
Let (M, ω) be a compact Kaehler n-manifold, and f any smooth function. **Then there exists a unique up to a constant function** φ such that $(\omega + dd^c\varphi)^n = Ae^f\omega^n$, where A is a positive constant obtained from the formula $\int_M Ae^f\omega^n = \int_M \omega^n$.

We are going to discuss a non-Kähler analogue of this result.

LCK manifolds

DEFINITION: A complex Hermitian manifold (M, I, g, ω) is called **locally conformally Kähler** (LCK) if there exists a closed 1-form θ such that $d\omega = \theta \wedge \omega$. The 1-form θ is called the **Lee form** and the g-dual vector field $\theta^\sharp$ is called the **Lee field**.

REMARK: This definition **is equivalent with the existence of a Kähler cover $(\tilde M, \tilde\omega) \to M$ such that the deck group Γ acts on $(M, \tilde\omega)$ by holomorphic homotheties.** Indeed, suppose that θ is exact, $df = \theta$. **Then $e^{-f}\omega$ is a Kähler form.**

THEOREM: (Vaisman)
A compact LCK manifold with non-exact Lee form **does not admit a Kähler structure.**

REMARK: Such manifold are called **strict LCK**. Further on, **we shall consider only strict LCK manifolds.**

Vaisman manifolds

DEFINITION: The LCK manifold (M, I, g, ω) is a **Vaisman manifold** if the Lee form is parallel with respect to the Levi-Civita connection.

THEOREM: A compact (strictly) LCK manifold M is Vaisman **if and only if it admits a non-trivial action of a complex Lie group of positive dimension,** acting by holomorphic isometries.

DEFINITION: **A linear Hopf manifold** is a quotient $M := \frac{\mathbb{C}^n \backslash 0}{\langle A \rangle}$ where A is a linear contraction. When A is diagonalizable, M is called **diagonal Hopf.**

CLAIM: **All diagonal Hopf manifolds are Vaisman,** and **all non-diagonal Hopf manifolds are LCK and not Vaisman**.

EXAMPLE: **Almost all non-Kähler compact complex surfaces are LCK.** Among those, **only elliptic surfaces and some Hopf surfaces are Vaisman.**

THEOREM: A compact complex manifold admits a Vaisman structure **if and only if it admits a holomorphic embedding to a diagonal Hopf manifold.**

Canonical foliation

REMARK: On a Vaisman manifold, the Lee and anti-Lee vector fields $\theta^\sharp$ and $I\theta^\sharp$ **are Killing and holomorphic.** Moreover, they commute: $[\theta^\sharp, I\theta^\sharp] = 0$. Therefore, **they define a holomorphic 1-dimensional foliation Σ.** This foliation is called **the canonical foliation.**

CLAIM: Let (M, ω, θ) be a Vaisman manifold, and $\theta^c := I(\theta($. Then the form $\omega_0 = -d\theta^c$ satisfies $\omega_0 = \omega - \theta \wedge \theta^c$. Moreover, this form is pseudo-Hermitian, invariant with respect to the action generated by $\theta^\sharp, I\theta^\sharp$, **vanishes on Σ and is positive in the transversal direction.**

COROLLARY: Clearly, a complex submanifold of an LCK manifold is LCK. Moreover, **a submanifold $Z \subset M$ of Vaisman manifold is Vaisman.**

Proof: Since ω_0 is exact, it satisfies $\int_Z \omega_0^{\dim_{\mathbb{C}} Z} = 0$. Therefore, **$Z$ is tangent to Σ everywhere.** This implies that Z is invariant under the Lie group action generated by $\theta^\sharp, I\theta^\sharp$. **A manifold admitting a holomorphic Killing action by a complex Lie group is Vaisman.** ■

Lee classes

Let (M,I) be a complex manifold admitting an LCK structure. In LCK geometry, the role of the Kähler cone is played by the **"Lee cone"**, i.e. **the set of classes in $H^1(M)$ which can be represented by Lee forms of an LCK structure**.

THEOREM: (Tsukada)
Let M be a compact Vaisman manifold. Then:

(i)

$$H^1(M) = H^{1,0}_d(M) \oplus \overline{H^{1,0}_d(M)} \oplus \langle\theta\rangle,$$

where $H^{1,0}_d(M)$ is the space of closed holomorphic 1-forms.

(ii) Consider a 1-form $\mu \in H^1(M)^*$ vanishing on $H^{1,0}_d(M) \oplus \overline{H^{1,0}_d(M)} \subset H^1(M)$ and satisfying $\mu([\theta]) > 0$. **Then a class $\alpha \in H^1(M,\mathbb{R})$ is a Lee class for some LCK structure if and only if $\mu(x) > 0$.** Moreover, **this LCK structure cam be chosen Vaisman**.

Lee field is determined by the complex structure

REMARK: Unlike the Lee class, which can be chosen in a half-space, **the Lee field on a Vaisman manifold is unique up to a constant multiplier.**

PROPOSITION: (Tsukada) Let M be a compact complex manifold of Vaisman type, and $\theta^\sharp$ the Lee field of a Vaisman structure (ω, θ). **Then $\theta^\sharp$ is determined by the complex structure on M uniquely up to a real multiplier.**

Proof: Consider the transversely Kähler form $\omega_0 = d\theta^c$. Its kernel is the canonical foliation $\Sigma = \langle\theta^\sharp, I(\theta^\sharp)\rangle$. Let ω_0' be a form associated in the same way with some other Vaisman structure (ω', θ') on M. Then $\eta := \omega_0 + \omega_0'$ is an exact, semi-positive (1,1)-form. This form cannot be strictly positive because $\int_M \eta^{\dim_{\mathbb C} M} = 0$, hence it has a non-trivial kernel, which is contained in $\ker\omega_0 \cap \ker\omega_0'$. However, $\dim_{\mathbb C}\ker\omega_0 = \dim_{\mathbb C}\ker\omega_0' = 1$, hence $\dim_{\mathbb C}(\ker\omega_0 \cap \ker\omega_0') = 1$. **This implies that the canonical foliation Σ' associated with (ω', θ') is equal to Σ.**

Step 2: Recall that $\theta^\sharp$ is holomorphic and Killing. Since Σ has a non-degenerate holomorphic section $\theta^\sharp$, it is trivial as a holomorphic line bundle. Since $\theta^\sharp$ is a holomorphic section of a trivial line bundle, **it is uniquely defined up to a real multiplier and a complex rotation.** ∎

The complex Monge-Ampère equation on LCK manifolds

DEFINITION: A form η on a Vaisman manifold **is Lee-invariant** if $\operatorname{Lie}_{\theta^\sharp}\eta = 0$ and **anti-Lee invariant** if $\operatorname{Lie}_{I\theta^\sharp}\eta = 0$.

The main result of today's talk.

THEOREM: Let (M,ω,θ) be a compact Vaisman manifold, and V' a Lee- and anti-Lee-invariant volume form on M, satisfying $\int_M V' = \int_M \omega^n$. **Then there exists a unique Vaisman metric ω' on M with the same Lee class and the volume form $(\omega')^n = V'$.**

Basic forms

DEFINITION: Let Σ be a foliation on a manifold M. **Basic forms** on (M, Σ) are differental forms on Σ which are locally obtained as pullbacks from the leaf space of the foliation.

CLAIM: A form η is Σ-basic **if and only if $i_X(\eta) = 0$ and $\operatorname{Lie}_X \eta = 0$ for any vector field $X \in \Sigma$.**

EXAMPLE: Let (M, I, ω, θ) be a Vaisman manifold. **Then the transversely Kähler form ω_0 is Σ-basic.**

Basic cohomology

DEFINITION: Clearly, de Rham differential The **basic cohomology** of (M, Σ) are de Rham cohomology of the complex of Σ-basic forms.

REMARK: Generally speaking, the basic cohomology of a compact manifold are hard to manage; in some very bad cases, **this space can be infinite-dimensional.** However, **if Σ is transversely Riemannian, the basic cohomology are finitely-dimensional, and can be computed using the Laplacian (P. Molino).**

THEOREM: (El Kacimi-Alaoui)
Let (M, I, ω, θ) be a compact Vaisman manifold of complex dimension n, and Σ its canonical foliation. Since Σ is transversely Kähler, the space of Σ-basic forms on M admits the Hodge decomposition and the standard Kähler identities. Moreover, **the basic cohomology can be represented by transversely harmonic forms which admit the Hodge decomposition, the Poincare duality and the Lefschetz $\mathfrak{sl}(2)$-action.** ■

Transversal Kähler form determines the Vaisman structure

LEMMA 1: Let M be a compact complex n-manifold of Vaisman type. **Then a Vaisman structure on M is uniquely determined by its transversal Kähler form ω_0 and the Lee class $[\theta] \in H^1(M, \mathbb{R})$.**

Proof: Since $\omega = d^c\theta + \theta\wedge\theta^c$, it suffices to show that the Lee form θ is uniquely determined by ω_0 and the Lee class. Let θ and θ' be two Lee forms of Vaisman structures, with the same transversal Kähler form ω_0. Denote by η the 1-form $\theta - \theta'$. Since $\omega_0 = d^c\theta = d^c\theta'$ this would imply $d^c\eta = d\eta = 0$. Such a 1-form cannot be exact, because if $\eta = df$, one has $dd^c f = 0$; however, pluriharmonic functions are constant on any compact manifold by the maximum principle. Therefore, θ cannot be cohomologous to θ'. ■

In other words, **the transversely Kähler form and the Lee class uniquely determine the Vaisman structure**.

Transversal Calabi-Yau theorem

THEOREM: (El Kacimi-Alaoui)
Let M be a compact Vaisman n-manifold, and Σ its canonical foliation. Then for any Σ-basic volume form V_0 which is basic cohomologous to an $(n-1)$-th power of a transversely Kähler form η_1, **there exists a unique transversely Kähler form η_2 in the same basic cohomology class such that $\eta_2^{n-1} = V_0$.**

Proof: "The proof is the same as for Kähler manifolds". ■

THEOREM: Let (M, ω, θ) be a compact Vaisman manifold, and V' a Lee- and anti-Lee-invariant volume form on M, satisfying $\int_M V' = \int_M \omega^n$. **Then there exists a unique Vaisman metric ω' on M with the same Lee class and the volume form $(\omega')^n = V'$.**

Proof: Clearly, $i_{I(\theta^\sharp)} i_{\theta^\sharp} \omega^n = n\omega_0^{n-1}$, where $\theta^\sharp$ is the Lee field. Since the Lee field is uniquely determined by the complex structure, the volume form is uniquely determined by the transversal volume form, and vice versa. Applying the transversal Calabi-Yau theorem, we find a bijective correspondence between the set of transversely Kähler structures on (M, Σ) and the set of volume forms (up to a constant multiplier). Now, Lemma 1 gives a bijective correspondence between the set of pairs [transversely Kähler structure, a Lee class] and the set of Vaisman structures. ■