Contraction loci in hyperkähler manifolds

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Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifol is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **of maximal holonomy** (also: simple, or IHS) if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.

Further on, all hyperkähler manifolds are assumed to be of maximal holonomy.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\begin{aligned}\lambda q(\eta,\eta) = &\int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ &- \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)\end{aligned}$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(3, b_2 - 3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

Extremal curves

DEFINITION: A rational curve on a complex variety is called **minimal** if for any two distinct points x, y on C, the space of deformations $Z_{x,y}$ of C passing through x, y is 0-dimensional.

REMARK: If $\dim Z_{x,y} > 0$, C can be deformed to a union of two curves, with one of them still passing through x, y. **Then it is not "minimal"** (in the usual sense: minimal curve is the one which cannot be deformed to a non-irreducible curve).

REMARK: Let C be a curve which can be deformed to a non-irreducible curve C'. On a compact Kähler manifold, degree of each components of C' is strictly smaller than degree of C, hence **any rational curve is cohomologous to a sum of minimal curves.**

REMARK: Using the BBF form, we shall identify $H_2(M, \mathbb{Q})$ and $H^2(M, \mathbb{Q})$. This allows us to consider the BBF form on $H_2(M, \mathbb{Q})$.

DEFINITION: A rational curve is called **extremal** if it is minimal, and its homology class has negative self-intersection.

Characterization of the Kähler cone

DEFINITION: The BBF form on $H^{1,1}(M,\mathbb{R})$ has signature $(1, b_2-2)$. This means that the set $\{\eta\in H^{1,1}(M,\mathbb{R}) \ \ | \ \ (\eta,\eta)>0\}$ has two connected components. The component which contains the Kähler cone Kah(M) is called **the positive cone**, denoted Pos(M).

THEOREM: (Huybrechts, Boucksom)
The Kähler cone of M is the set of all $\eta\in \mathrm{Pos}(M)$ such that $(\eta, C)>0$ for all extremal curves C.

In other words, **the Kähler cone is locally polyhedral** (with some round pieces in the boundary), and its faces are orthogonal complements to the extremal curves.

MBM classes

REMARK: Suppose that M and M' are two birational Calabi-Yau manifolds (e. g. holomorphically symplectic manifolds). **Then $H^2(M)$ is naturally identified with $H^2(M')$.** Indeed, M' is obtained from M by a sequence of blow-ups and blow-downs, but since their canonical bundles are trivial, all blown up divisors are blown down in the end, and M is identified with M' outside of real codimension 4.

THEOREM: (Ekaterina Amerik, V.) Let η be a cohomology class of an extremal curve on a hyperkähler manifold, and (M_1, η) be obtained as a deformation of (M, η) in a continuous family such that η remains of type $(1,1)$ for all fibers of this family. **Then M_1 is birational to a hyperkähler manifold M_1' such that η is a class of an extremal curve on M_1'.**

DEFINITION: A class $\eta \in H^2(M, \mathbb{Z})$ which can be represented by an extremal curve for some complex holomorphically symplectic structure on M is called **an MBM class**.

REMARK: Equivalent definition: **An MBM class $\eta \in H^2(M)$ is a class which can be represented by a rational curve in (M, I) when (M, I) is a non-algebraic deformation of M with $\mathrm{Pic}(M, I)_{\mathbb{Q}} = \langle \eta \rangle$.**

Kawamata bpf

DEFINITION: Base point set of a holomorphic line bundle is an intersection of all zero divisors of all sections of its tensor powers. A line bundle with trivial base point set is called **base point free** (bpf). A line bundle L with nL bpf is called **semiample**.

CLAIM: Let L be a semiample line bundle on a compact complex variety M. **Then M is equipped with a holomorphic map $\varphi: M \longrightarrow X$ such that $L = \varphi^* L_0$, where L_0 is an ample bundle on X.**

DEFINITION: A line bundle L is **nef** if $c_1(L)$ lies in the closure of the Kähler cone, and **big** if $H^0(M, L^N) = O(\dim M^N)$.

THEOREM: (Kawamata bpf theorem; very weak form)
Let L be a nef line bundle on M such that $nL - K_M$ is big. **Then L is semiample.**

For Calabi-Yau manifolds this means just that **big and nef bundles are semiample.**

Birational contractions

DEFINITION: Birational contraction of a complex manifold is a holomorphic birational map $M \longrightarrow X$ to a complex variety X.

REMARK: From Kawamata bpf it follows that **any big and nef bundle L on Calabi-Yau is obtained as $L = \varphi^* L_0$, where $\varphi: M \longrightarrow X$ is a birational contraction and L_0 an ample bundle on X.**

DEFINITION: A variety is called **rationally connected** if any two of its points can be connected by a sequence of rational curves

THEOREM: Let $\varphi: M \longrightarrow X$ be a birational contraction of a Calabi-Yau manifold. **Then any fiber $\varphi^{-1}(x)$ is rationally connected.**
Proof: Highly non-trivial (Kawamata, Shokurov, Hacon-McKernan). ■

REMARK: Let M be a projective hyperkähler manifold, η the cohomology class of an extremal curve, ω_0 an integer point on the corresponding face of the Kähler cone, and L the holomorphic line bundle with $c_1(L) = \omega_0$. Then L is big (by Grauert-Riemenschneider conjecture, proven by Siu and Demailly) and nef. The corresponding birational contraction **contracts all curves C with $[C] = \lambda\eta$.** Indeed, $\langle L_1, C\rangle = 0$.

Faces of the Kähler cone

DEFINITION: Let M be a hyperkähler manifold. **A codimension 1 face**, or **a face** of the Kähler cone is a subset of its boundary obtained as an intersection of this boundary and a hyperplane which has dimension $h^{1,1}-1$.

THEOREM: Codimension 1 faces of a Kähler cone **are in bijective correspondence with cohomology classes η of extremal curves.** Each such face is obtained as an intersection of the boundary and $\eta^\perp$.

THEOREM: Let (M,I) be a hyperkähler manifold and S the set of all MBM classes of type (1,1) on (M,I). Let $S^\perp$ the union of all orthogonal complement to all $s \in S$. Then Kah(M,I) is a connected component of Pos$(M)\backslash S^\perp$, and **each connected component of Pos$(M)\backslash S^\perp$ can be realized as a Kähler cone of some birational model of (M,I).**

Proof: Follows from the deformational stability of MBM curves and the global Torelli theorem. ■

Centers of birational contraction are homeomorphic

REMARK: Clearly, $H^{1,1}(M)$ is obtained as orthogonal complement to the 2-dimensional space $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega\rangle$, where Ω is the cohomology class of the holomorphic symplectic form. Then $\operatorname{Pic}(M) = H^{1,1}(M) \cap H^2(M, \mathbb{Z})$ has maximal rank only if the plane $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega\rangle$ is rational. **There is at most countable number of such M.**

THEOREM: (Ekaterina Amerik, V.) Let F be a face of a Kähler cone of a hyperkähler manifold, C the corresponding rational curve, and (M_1, F_1) be obtained as a deformation of (M, F) in a continuous family such that the cohomology class $[C]$ remains of type (1,1). Assume that neither M nor M_1 has maximal Picard rank, and $b_2(M) - k > 3$. **Then there exists a homeomorphism $\Psi: M \longrightarrow M_1$ identifying the corresponding contraction centers and the contracted extremal curves.**

REMARK: Existence of a homeomorphism follows from ergodicity of mapping group action, global Torelli theorem, and Thom-Mathers stratification of proper real analytic maps.

REMARK: Stability of minimal rational curves under deformations of hyperkähler manifolds is in essentially due to Ziv Ran and Claire Voisin: **if you deform a hyperkähler manifold with a minimal rational curve, and its cohomology class remains of type (1,1), the curve also deforms.**

Centers of birational contraction for $K3^{[2]}$

The homeomorphism theorem allows one to give a classification of birational contraction centers in terms of the period spaces and lattices.

THEOREM: Let M be a deformation of $K3^{[2]}$, and Z a birational contraction center associated with a face of a Kähler cone. **Then one of the following three cases occurs:**

(a) Z is Lagrangian $\mathbb{C}P^2$ obtained as a deformation of $C^{[2]} \subset M^{[2]}$, where $C \subset M$ is a smooth -2-curve on a K3 surface M.

(b) Z is a deformation of the exceptional divisor on $M^{[2]}$.

(c) Z is a singular divisor obtained as a deformation of $Z_C \subset M^{[2]}$, where Z_C is the set of all length 2 ideals on M with support intersecting C.

In all three cases Z is homeomorphic to its model in $M^{[2]}$.

Centers of birational contraction for K3$^{[2]}$ and lattices

This result is implied by the following lattice-theoretic result.

THEOREM 1: Let M be a K3 and Λ be the lattice $H^2(M^{[2]}, \mathbb{Z})$ with its BBF form. Denote by Γ the group of isometries of Λ generated by reflections $x \longrightarrow x - 2\frac{(x,z)}{(z,z)} z$ with negative z **("negative reflections")** and let $E \in \Lambda$ be the exceptional divisor of E. **Then for any $x \in \Lambda$ with $(x,x) < 0$, there exists $\gamma \in \Gamma$ such that the rank 2 lattice $\langle \gamma(x), E \rangle$ has no positive vectors.**

Proof: Follows from the classification of orbits of Γ on Λ, due to Gritsenko, Hulek, Sankaran: they prove that there are at most 2 orbits of Γ action on the set $\{x \in \Lambda \mid (x,x) = w\}$ for any given w. One of these orbits intersects $H^2(M) \subset \Lambda$ and for such an orbit Theorem is obvious. Another orbit is E and then it starts from $w \leqslant -8$. The later contains a vector $\gamma(x) = E + y$ where $y \in H^2(M)$, and $\langle \gamma(x), E \rangle$ is negative definite. ■

CONJECTURE: This result is true for $M^{[n]}$ for all n.

REMARK: If this is true, we have a similar simple classification of contraction centers for all deformations of Hilbert schemes on K3.

Teichmüller spaces

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let $\operatorname{Teich} := \operatorname{Comp}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

REMARK: In all known cases Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

THEOREM: (Bogomolov-Tian-Todorov) Teich **is a complex manifold when M is Calabi-Yau**.

Definition: Let $\operatorname{Diff}(M)$ be the group of diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ **the mapping class group**.

REMARK: The quotient $\operatorname{Teich}/\Gamma$ **is identified with the set of equivalence classes of complex structures.**

The period map

Remark: For any $J \in \operatorname{Teich}$, (M,J) is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.

Definition: Let $\operatorname{Per}: \operatorname{Teich} \longrightarrow {\mathbb P}H^2(M,{\mathbb C})$ map J to a line $H^{2,0}(M,J) \in {\mathbb P}H^2(M,{\mathbb C})$. The map $\operatorname{Per}: \operatorname{Teich} \longrightarrow {\mathbb P}H^2(M,{\mathbb C})$ is called **the period map**.

REMARK: **Per maps Teich into an open subset of a quadric,** defined by

$$\mathbb{P}\mathrm{er} := \{l \in {\mathbb P}H^2(M,{\mathbb C}) \ \ | \ \ q(l,l)=0, q(l,\bar l) > 0\}.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1) = \operatorname{Gr}_{++}(H^2(M,{\mathbb R}))$. Indeed, the group $SO(H^2(M,{\mathbb R}),q) = SO(b_2-3,3)$ acts transitively on $\mathbb{P}\mathrm{er}$, and $SO(2)\times SO(b_2-3,1)$ is a stabilizer of a point.

Global Torelli theorem

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts) Two points $I, I' \in$ Teich **are non-separable if and only if there exists a bimeromorphism $(M, I) \longrightarrow (M, I')$ which is non-singular in codimension 2 and acts as identity on $H^2(M)$.**

REMARK: This is possible only if (M, I) and (M, I') contain a rational curve. **General hyperkähler manifold has no curves;** ones which have curves belong to a countable union of divisors in Teich.

DEFINITION: The space $\operatorname{Teich}_b :=$ Teich $/ \sim$ is called **the birational Teichmüller space** of M. Since Teich_b is obtained by gluing together all non-separable points, it is also called **Hausdorff reduction** of Teich.

THEOREM: **(Torelli theorem for hyperkähler manifolds)**
The period map $\operatorname{Teich}_b \xrightarrow{\mathsf{Per}} \mathbb{P}\mathrm{er}$ is a diffeomorphism, for each connected component of Teich_b.

Mapping class group for Hilbert scheme of a K3

THEOREM: (E. Markman) Let X be a deformation of the Hilbert scheme $M^{[n]}$ of K3, Γ a subgroup of its mapping class group fixing a connected component of Teich_b, and $\Lambda = H^2(X, \mathbb{Z})$ with its BBF form. **Then Γ is the subgroup of $O(\Lambda)$ generated by negative reflections.**

THEOREM: Let $Z \subset X$ be a birational contraction center on a deformation X of a Hilbert scheme $M^{[n]}$ of K3. Suppose that Theorem 1 is true for $M^{[n]}$. **Then the pair (X, Z) can be smoothly deformed to $(M^{[n]}, Z')$, where M is a non-algebraic K3.**

Proof: Let η be the MBM class associated with Z and $\operatorname{Teich}_\eta$ the Teichmüller space of deformations of X such that η remains of type (1,1). Then any lattice $\Lambda_1 \subset \Lambda$ with $\Lambda_1^\perp$ containing a positive 2-plane and $\Lambda_1 \ni \eta$ can be realized as a Picard lattice of $I \in \operatorname{Teich}_\eta$. Applying this to $\langle \gamma(\eta), E\rangle$ from Theorem 1, we obtain a deformation of (X, Z) with a Picard lattice $\langle \gamma(\eta), E\rangle$, and this is a Hilbert scheme for a non-algebraic K3. ■

REMARK: All curves on a Hilbert scheme $M^{[n]}$ of a non-algebraic K3 M can be contracted. This contraction gives a symmetric power of a singular K3 which has no curves at all. This gives an explicit description of aqll contraction centers on $M^{[n]}$.