Birational contractions of hyperkaehler manifolds are diffeomorphic

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Seminar of Laboratory of Algebraic Geometry

Moscow, HSE, July 22, 2016

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(3, b_2 - 3)$.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

Extremal curves

DEFINITION: A rational curve on a complex variety is called **minimal** if for any two distinct points x, y on C, the space of deformations $Z_{x,y}$ of C passing through x, y is 0-dimensional.

REMARK: If $\dim Z_{x,y} > 0$, C can be deformed to a union of two curves, with one of them still passing through x, y. Then it is not "minimal" (in the usual sense: minimal curve is one which cannot be deformed to a non-irreducible curve).

REMARK: Let C be a curve which can be deformed to a non-irreducible curve C'. On a compact Kähler manifold, degree of each components of C' is strictly smaller than degree of C, hence **any curve is cohomologous to a sum of minimal curves.**

REMARK: Using the BBF form, we shall identify $H_2(M,\mathbb{Q})$ and $H^2(M,\mathbb{Q})$. This allows us to consider the BBF form on $H_2(M,\mathbb{Q})$.

DEFINITION: A rational curve is called **extremal** if it is minimal, and its homology class has negative self-intersection.

Characterization of the Kähler cone

DEFINITION: The BBF form on $H^{1,1}(M,\mathbb{R})$ has signature $(1, b_2 - 2)$. This means that the set $\{\eta \in H^{1,1}(M,\mathbb{R}) \ \ | \ \ (\eta,\eta) > 0\}$ has two connected components. The component which contains the Kähler cone $\operatorname{Kah}(M)$ is called **the positive cone**, denoted $\operatorname{Pos}(M)$.

THEOREM: (Huybrechts, Boucksom)
The Kähler cone of M is the set of all $\eta \in \operatorname{Pos}(M)$ such that $(\eta, C) > 0$ for all extremal curves C.

In other words, Kähler cone is locally polyhedral (with some round pieces in the boundary), and its faces are orthogonal complements to the extremal curves.

This result follows from Kawamata base point free theorem (see below).

Birational contractions and Kawamata bpf

DEFINITION: Base point set of a holomorphic line bundle is an intersection of all zero divisors of its sections. A line bundle with trivial base point set is called **base point free** (bpf). A line bundle L with nL bpf is called **semiample**

CLAIM: Let L be a semiample line bundle on a compact complex variety M. **Then M is equipped with a holomorphic map $\varphi: M \longrightarrow X$ such that $L = \varphi^* L_0$, where L_0 is an ample bundle on X.**

DEFINITION: A line bundle L is **nef** if $c_1(L)$ lies in the closure of the Kähler cone, and **big** if $\int_M c_1(L)^{\dim_{\mathbb{C}} M} > 0$.

THEOREM: (Kawamata bpf theorem; very weak form)
Let L be a nef line bundle on M such that $nL - K_M$ is big. **Then L is semiample.**

For Calabi-Yau manifolds this means just that **big and nef bundles are semiample.**

Birational contractions

DEFINITION: Birational contraction of a complex manifold is a holomorphic birational map $M \longrightarrow X$ to a complex variety X.

REMARK: From Kawamata bpf it follows that **any big and nef bundle L on Calabi-Yau is obtained as $L = \varphi^* L_0$, where $\varphi: M \longrightarrow X$ is a birational contraction and L_0 an ample bundle on X.**

DEFINITION: A variety is called **rationally connected** if any two of its points can be connected by a sequence of rational curves

DEFINITION: A **Calabi-Yau manifold** is a compact, Kähler manifold M with $c_1(M) = 0$.

Theorem 1: Let $\varphi: M \longrightarrow X$ be a birational contraction of a Calabi-yau manifold. **Then any fiber $\varphi^{-1}(x)$ is rationally connected.**
Proof: Highly non-trivial (Shokurov and others). ■

REMARK: Let M be a hyperkähler manifold, η the cohomology class of an extremal curve, ω_0 an integer point on the corresponding face of the Kähler cone, and L the holomorphic line bundle with $c_q(L) = \omega_0$. Then L is big and nef. Then the corresponding birational contraction **contracts all curves C with $[C] = \lambda\eta$.** Indeed, $\langle L_1, C\rangle = 0$.

MBM classes

REMARK: Suppose that M and M' are two birational Calabi-Yau manifolds (e. g. holomorphically symplectic manifolds). **Then $H^2(M)$ is naturally identified with $H^2(M')$.** Indeed, M' is obtained from M by a sequence of blow-ups and blow-downs, but since their canonical bundles are trivial, all blown up divisors are blown down in the end, and M is identified with M' outside of real codimension 4.

THEOREM: (Ekaterina Amerik, V.) Let η be a cohomology class of an extremal curve on a hyperkähler manifold, and (M_1, η) be obtained as a deformation of (M, η) in a continuous family such that η remains of type $(1,1)$ for all fibers of this family. **Then M_1 is birational to a hyperkähler manifold M_1' such that η is a class of extremal curve on M_1'.**

DEFINITION: A class $\eta \in H^2(M, \mathbb{Z})$ which can be represented by an extremal curve for some complex holomorphically symplectic structure on M is called **an MBM class**.

Faces of the Kähler cone

DEFINITION: Let M be a hyperkähler manifold. **A codimension 1 face**, or (sometimes) just **a face** of the Kähler cone is a subset of its boundary obtained as an intersection of this boundary and a hyperplane which has dimension $h^{1,1}-1$. **A face of codimension k** of the Kähler cone is an intersection of k adjacent codimension 1 faces.

THEOREM: (Huybrechts, Boucksom) Codimension 1 faces of a Kähler cone **are in bijective correspondence with cohomology classes η of extremal curves.** Each such face is obtained as in intersection of the boundary and $\eta^\perp$.

Faces of the Kähler cone and birational contractions

THEOREM: Let M be a projective hyperkähler manifold, and $\pi : M \longrightarrow M_1$ a birational contraction. Consider the set $[C_1], ..., [C_k]$ of cohomology classes of all extremal curves which are contracted by π. **Then $\bigcap_i [C_i]^\perp \cap \partial \operatorname{Kah}$ is a codimension k face of the Kähler cone.** Moreover, **faces are in bijective correspondence with birational contractions.**

Proof: Let L_1 be an ample bundle on M_1. Then $L := \pi^* L_1$ is a big, nef bundle with $c_1(L) \in \bigcap_i [C_i]^\perp$, hence the set $\bigcap_i [C_i]^\perp$ is a non-empty face. Conversely, for any face $F := \bigcap_i [C_i]^\perp \cap \partial \operatorname{Kah}$, and any line bundle with $c_1(L)$ in interior of F, the bundle L is big and nef and the corresponding contraction contracts curves in $[C_i]$ and only them. ∎

REMARK: Define **nef cone** of a projective variety as the set of all (1,1)-classes which are non-negative on curves. **Then the nef cone of M_1 is identified with the interior of the face F.**

Centers of birational contraction are diffeomorphic

REMARK: Clearly, $H^{1,1}(M)$ is obtained as orthogonal complement to the 2-dimensional space $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega\rangle$, where Ω is the cohomology class of the holomorphic symplectic form. Then $\operatorname{Pic}(M) = H^{1,1}(M) \cap H^2(M,\mathbb{Z})$ has maximal rank only if the plane $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega\rangle$ is rational. **There is at most countable number of such M.**

The main result of this talk:

THEOREM: (Ekaterina Amerik, V.) Let F be a codimension k face of a Kähler cone of a hyperkähler manifold, $F = \bigcap [C_i]^\perp \cap \partial \operatorname{Kah}$, and (M_1, F_1) be obtained as a deformation of (M, F) in a continuous family such that all $[C_i]$ remain of type (1,1). Assume that neither M nor M_1 has maximal Picard rank, and $b_2(M) - k > 3$. **Then there exists a diffeomorphism $\Psi: M \longrightarrow M_1$ identifying the corresponding contraction centers and the contracted extremal curves.**

REMARK: Stability of minimal rational curves under deformations of hyperkähler manifolds is in essentially due to Ziv Ran and Claire Voisin: **if you deform a hyperkähler manifold with a minimal rational curve, and its cohomology class remains of type (1,1), the curve also deforms.**

Teichmüller spaces

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let $\operatorname{Teich} := \operatorname{Comp}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

REMARK: In all known cases Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

THEOREM: (Bogomolov-Tian-Todorov) Teich **is a complex manifold when M is Calabi-Yau**.

Definition: Let $\operatorname{Diff}(M)$ be the group of diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ **the mapping class group**.

REMARK: The quotient $\operatorname{Teich}/\Gamma$ **is identified with the set of equivalence classes of complex structures.**

Computation of the mapping class group

THEOREM: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

THEOREM: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\Big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

REMARK: Sullivan's theorem implies that the mapping class group for $\dim_{\mathbb{C}} M \geqslant 3$, $\pi_1(M) = 0$, **is an arithmetic lattice**. Very much unlike the mapping class group for curves!

The period map

Remark: For any $J \in \operatorname{Teich}$, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $\operatorname{Per} : \operatorname{Teich} \longrightarrow {\mathbb P}H^2(M, {\mathbb C})$ map J to a line $H^{2,0}(M, J) \in {\mathbb P}H^2(M, {\mathbb C})$. The map $\operatorname{Per} : \operatorname{Teich} \longrightarrow {\mathbb P}H^2(M, {\mathbb C})$ is called **the period map**.

REMARK: **Per maps Teich into an open subset of a quadric,** defined by

$$\mathbb{P}\mathrm{er} := \{l \in {\mathbb P}H^2(M, {\mathbb C}) \ \ | \ \ q(l, l) = 0, q(l, \bar l) > 0\}.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1) = \operatorname{Gr}_{++}(H^2(M, {\mathbb R}))$**.** Indeed, the group $SO(H^2(M, {\mathbb R}), q) = SO(b_2 - 3, 3)$ acts transitively on $\mathbb{P}\mathrm{er}$, and $SO(2) \times SO(b_2 - 3, 1)$ is a stabilizer of a point (see below for a more detailed argument).

THEOREM: (Bogomolov) For any hyperkähler manifold, **period map is locally a diffeomorphism.**

Period space as a Grassmannian of positive 2-planes

PROPOSITION: The period space

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l, \bar{l}) > 0\}.$$

is identified with $SO(b_2-3,3)/SO(2) \times SO(b_2-3,1)$**,** which is a Grassmannian of positive oriented 2-planes in $H^2(M, \mathbb{R})$.

Proof. Step 1: Given $l \in \mathbb{P}H^2(M, \mathbb{C})$, **the space generated by** $\operatorname{Im} l, \operatorname{Re} l$ **is 2-dimensional,** because $q(l,l) = 0, q(l, \bar{l})$ implies that $l \cap H^2(M, \mathbb{R}) = 0$.

Step 2: This 2-dimensional plane is positive, because $q(\operatorname{Re} l, \operatorname{Re} l) = q(l + \bar{l}, l + \bar{l}) = 2q(l, \bar{l}) > 0$.

Step 3: Conversely, for any 2-dimensional positive plane $V \in H^2(M, \mathbb{R})$, **the quadric** $\{l \in V \otimes_{\mathbb{R}} \mathbb{C} \ \ | \ \ q(l,l) = 0\}$ **consists of two lines;** a choice of a line is determined by orientation. ■

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts) Two points $I, I' \in \text{Teich}$ **are non-separable if and only if there exists a bimeromorphism $(M,I) \longrightarrow (M,I')$ which is non-singular in codimension 2 and acts as identity on $H^2(M)$.**

REMARK: This is possible only if (M,I) and (M,I') contain a rational curve. **General hyperkähler manifold has no curves;** ones which have curves belong to a countable union of divisors in Teich.

DEFINITION: The space $\text{Teich}_b := \text{Teich}/\sim$ is called **the birational Teichmüller space** of M. Since Teich_b is obtained by gluing together all non-separable points, it is also called **Hausdorff reduction** of Teich,

THEOREM: **(Torelli theorem for hyperkähler manifolds)**
The period map $\text{Teich}_b \xrightarrow{\text{Per}} \mathbb{P}\text{er}$ is a diffeomorphism, for each connected component of Teich_b.

Ergodic complex structures

DEFINITION: Let M be a complex manifold, Teich its Techmüller space, and Γ the mapping group acting on Teich. **An ergodic complex structure** is a complex structure with dense Γ-orbit.

DEFINITION: Let (M, μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore, the set Z_U of such orbits has measure 0.

Step 2: Choose a countable base $\{U_i\}$ of topology on M. Then the set of points in dense orbits is $M \backslash \bigcup_i Z_{U_i}$. ■

Ergodicity of the mapping class group action

DEFINITION: A lattice in a Lie group is a discrete subgroup $\Gamma \subset G$ such that G/Γ has finite volume with respect to Haar measure.

THEOREM: (Calvin C. Moore, 1966) Let Γ be a lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact subgroup. **Then the left action of Γ on G/H is ergodic.**

THEOREM: Let $\mathbb{P}\mathrm{er}$ be a component of a birational Teichmüller space, and Γ its monodromy group. Let $\mathbb{P}\mathrm{er}_e$ be a set of all points $L \subset \mathbb{P}\mathrm{er}$ such that the orbit $\Gamma \cdot L$ is dense. **Then $Z := \mathbb{P}\mathrm{er} \setminus \mathbb{P}\mathrm{er}_e$ has measure 0.**

Proof. Step 1: Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. **Then Γ-action on G/H is ergodic,** by Moore's theorem.

Step 2: Ergodic orbits are dense, becuse the union of non-ergodic orbits has measure 0. ■

THEOREM: Let (M, I) be a hyperkähler manifold with $b_2 > 3$ or a compact torus of dimension > 1. **Then I ergodic (has dense Γ-orbit in Teich) if and only if $\operatorname{rk}\operatorname{Pic}(M, I)$ is not maximal.**

Proof: Follows from M. Ratner's theorems. ■

The space Teich_F

DEFINITION: Let M be a hyperkähler manifold, $F := \bigcap_i [C_i]^\perp \cap \partial \operatorname{Kah}$ a face of the Kähler cone, and Teich_F denote the Teichmüller space of all deformations of M such that all C_i remain extremal. Denote by $\mathbb{P}\mathrm{er}_F$ the corresponding period space,

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l,\bar{l}) > 0, q(l,[C_i]) = 0\}.$$

Clearly, $\mathbb{P}\mathrm{er}_F = \mathrm{Gr}_{++}(\bigcap_i [C_i]^\perp)$.

REMARK: Global Torelli theorem and stability of extremal curves under deformations imply that $\mathrm{Per}: \ \operatorname{Teich}_F \longrightarrow \mathbb{P}\mathrm{er}_F$ **is the Hausdorff reduction map (gluing together all non-separable points).**

Then the same Ratner theorem argument as above gives the following result.

THEOREM: Let F be a face of the Kähler cone, and Γ_F be a subgroup of the mapping class group Γ preserving the connected component of Teich_F. **Then an orbit $\Gamma_F \cdot I$ of Γ_F on Teich_F is dense if and only if $b_2(M) - \operatorname{codim} F > 3$ and (M,I) has non-maximal Picard lattice.**

The main result deduced from ergodicity

THEOREM: (Whitney)
Let $\mathcal{X} \supset \mathcal{M} \xrightarrow{\Psi} B$ be a holomorphic family of pairs $X \subset M$ of compact complex varieties, with M smooth. Consider the set $B_0 \subset B$ of all points $b \in B$ such that the family Ψ admits a smooth trivialization in a neighbourhood of b, in such a way that all fibers of $\Psi|_{\mathcal{X}}$ are identified. **Then B_0 is open in B.**

THEOREM: Let $F = \bigcap [C_i]^\perp \cap \partial \mathrm{Kah}$ be a face of the Kähler cone of a hyperkähler manifold, and (M_1, F) be obtained as a deformation of (M, F) in a continuous family such that all C_i remain of type $(1,1)$ for all fibers of this family, and F is a face on M_1. Assume that neither M nor M_1 has maximal Picard rank. **Then there exists a diffeomorphism $\Psi : M \longrightarrow M_1$ identifying the centers of corresponding birational contractions, and minimal curves in the cogomology classes $[C_i]$.**

Proof: Let $\mathcal{U} \longrightarrow \mathrm{Teich}_F$ be the corresponding universal family over the Teichmüller space, and $\mathrm{Teich}_F^0 \subset \mathrm{Teich}_F$ the subset consisting of all points I such that the universal family admits a smooth trivialization in a neighbourhood of I compatible with centers of contraction as in Whitley theorem. Since Teich_F^0 is non-empty, open and Γ_F-invariant, it contains all dense orbits. ■