Locally trivial families in real analytic geometry

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12th Mini Workshop on Singularities, Geometry and Differential Equations

and 1st Meeting on Foliations and Singularities,

Febrary 1, 2019

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Joint work with Ekaterina Amerik

Teichmüller spaces

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ the connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let $\operatorname{Teich} := \operatorname{Comp}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

REMARK: In almost all known cases Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}(M)$ be the group of diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ **the mapping class group**.

REMARK: The quotient $\operatorname{Teich}/\Gamma$ **is identified with the set of equivalence classes of complex structures.**

THEOREM: For hyperkähler manifolds and complex tori T, $\dim_{\mathbb{C}} T > 1$, **the mapping class group Γ acts on Teich with dense orbits.**

Families with a discrete group acting with dense orbits

THEOREM: Let $\pi : \mathcal{X} \longrightarrow B$ be a family of complex varietiees, and Γ a group which acts on B with all orbits dense. **Then the fibers of π are Lipschitz homeomorphic, and these homeomorphisms are smooth in the strata of Whitney stratification.**

Proof: Follows from Thom-Mather theory (see the lectures on equisingularity by Terence Gaffney in this meeting). Indeed, any such family is equisingular in a Zariski open subset $B_0 \subset B$, and since B_0 is Γ-invariant, we have $B = B_0$.

QUESTION: Can we do better?

REMARK: In complex geometry - no, we cannot. Indeed, there is a dense subset in the "marked moduli" (Teichmüller space) of hyperkähler manifolds or complex tori T, $\dim_{\mathbb{C}} T > 1$, where the mapping class group acts with dense orbits.

For "local triviality" to work, we need to find a category without continuous moduli of deformations.

Continuous moduli of deformations in real analytic category

Real analytic manifolds do not have continuous moduli: indeed, their deformations are controlled by the first cohomology of the tangent bundle, and higher cohomologies of a coherent sheaf over a real analytic manifold always vanish.

However, **real analytic varieties have continuous moduli of deformations**. The "four lines in $\mathbb{R}P^2$" example was already mentioned in lectures by Terence Gaffney:

Let C be a configuration of 4 real lines in $\mathbb{R}P^2$. If these lines intersect in one point, the corresponding tangent cone (which is determined intrinsically by the real analytic geometry of the pair $(\mathbb{R}P^2, C)$) is 4 lines in a vector space. **The cross-ratio of these 4 lines gives a continuous real analytic invariant of this pair.**

Locally trivial deformations (Flenner, Kosarew)

DEFINITION: (Flenner, Kosarew)
Let $\pi : \mathcal{X} \longrightarrow B$ be a family of complex varieties. Assume that any point $x \in \mathcal{X}$ has a neighbourhood W which is biholomorphic to a product $F \times U$ such that $\pi\big|_{F\times U}$ is a projection to U. Then π is called **a locally trivial deformation**.

In real analytic category, such deformations have no continuous moduli, see below.

THEOREM: (Namikawa)
Every flat deformation of a projective holomorphically symplectic variety with $\mathbb{Q}$-factorial terminal holomorphically symplectic singularities **is locally trivial.**

REMARK: A similar result by Namikawa **holds for canonical singularities** (see Bakker-Lehn theorem below).

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot,\cdot)$, $\omega_J := g(J\cdot,\cdot)$, $\omega_K := g(K\cdot,\cdot)$.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: **(Calabi-Yau)** A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

Bogomolov decomposition

DEFINITION: A hyperkähler manifold M is called **of maximal holonomy** (also: simple, or IHS) if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.

Further on, all hyperkähler manifolds are assumed to be of maximal holonomy.

Birational contractions and locally trivial deformations

DEFINITION: Birational contraction of a complex manifold is a holomorphic birational map $M \longrightarrow X$ to a complex variety X.

THEOREM: (Bakker-Lehn)
Let $f : M \longrightarrow M_1$ be a birational contraction of a projective hyperkähler manifold, with $b_2(M_1) \geqslant 5$. Let $\operatorname{Def}^{lt}(M_1) \subset \operatorname{Def}(M_1)$ be the subspace parametrizing locally trivial deformations of M_1 and $\operatorname{Def}(M, f) \subset \operatorname{Def}(M)$ be the subspace of deformations of the pair (M, f). Assume that $\dim \operatorname{Def}(M_1) \geqslant 2$. Then the contraction **induces an isomorphism between** $\operatorname{Def}(M, f)$ **and** $\operatorname{Def}^{lt}(M_1)$**,** so that **the small deformations of** (M, f) **map isomorphically onto "locally trivial" small deformations of** M_1**.**

Proof: The proof is based on ergodic properties of the mapping class group action and results of Y. Namikawa on local triviality of holomorphic symplectic deformations.

Birational contractions are real analytically identified

THEOREM: (Amerik-V.)
Let M be a hyperkähler manifold, Pic(M) not maximal, and $f : M \longrightarrow M_1$ a birational contraction. Assume that the space of deformations of the map $f : M \longrightarrow M_1$ has dimension $\geqslant 2$. **Then the corresponding contraction loci are real analytically equivalent.**

Proof: We use two ingredients: ergodic action of the mapping class group on the corresponding Teichmüller space and real analytic triviality of locally trivial deformations.

Locally trivial deformations are real analytically trivial

THEOREM: Let $\pi : \mathcal{X} \longrightarrow B$ be a deformation of complex varieties, which is locally trivial. **Then the real analytic map $\pi_{\mathbb{R}} : \mathcal{X}_{\mathbb{R}} \longrightarrow B_{\mathbb{R}}$ underlying π defines a family which is trivial over any sufficiently small open set $U \subset B$.**

Proof. Step 1: By Artin's analytification theorem it would suffice to trivialize the family $\pi_{\mathbb{R}}$ in a formal neighbourhood $\hat{F}$ of $F := \pi^{-1}(b)$, for all $b \in B$. Locally in $\mathcal{X}$, the complex family π is a product. The local-in-$\mathcal{X}$ trivialization of π defines a Čech cocycle $w \in H^1(F, \operatorname{Aut}_F(\hat{F}))$ where $\operatorname{Aut}_F(\hat{F}))$ is the group sheaf of formal automorphisms of $\hat{F}$ trivial on $F \subset \hat{F}$ and commuting with the projection to B.

Step 2: The sheaf $\operatorname{Aut}_F(\hat{F}))$ can be obtained as a limit of sheaves of automorphisms of infinitesimal neighbourhood $F_k \subset \hat{F}$ of order k. Therefore, $w \in H^1(F, \operatorname{Aut}(\hat{F}))$ vanishes whenever its finite order representatives $w_k \in H^1(F, \operatorname{Aut}_F(F_k))$ vanish.

Locally trivial deformations are real analytically trivial (2)

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Step 3: The Lie groups $\operatorname{Aut}_F(F_k))$ are nilpotent, and fit into exact sequences

$$0 \longrightarrow V_k \longrightarrow \operatorname{Aut}_F(F_k) \longrightarrow \operatorname{Aut}_F(F_{k-1}) \longrightarrow 0$$

where V_k is a sheaf of abelian unipotent groups, that is, a coherent sheaf. In the corresponding exact sequence of first cohomology

$$H^1(V_k) \longrightarrow H^1(\operatorname{Aut}_F(F_k)) \longrightarrow H^1(\operatorname{Aut}_F(F_{k-1})) \longrightarrow H^2(V_k)$$

all terms vanish, because higher cohomology of any coherent sheaf on a real analytic variety vanishes (Cartan), hence $H^1(V_k) = H^2(V_k) = 0$.

Step 4: We obtain that the group sheaf $\operatorname{Aut}_F(F_k)$ is filtered by normal subgroups with coherent subquotients, hence has vanishing cohomology. ■

Computation of the mapping class group

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$,** and $c > 0$ a rational number.

DEFINITION: The form q is called **Bogomolov-Beauville-Fujiki form**. It has signature $(3, b_2 - 3)$.

THEOREM: (V., 1996, 2009) Let M be a maximal holonomy, compact hyperkähler manifold, and $\Gamma_0 = \operatorname{Aut}(H^*(M,\mathbb{Z}), p_1, ..., p_n)$. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}), q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}), q)$ **has finite kernel.**
(iii) The tautological map $\Gamma \longrightarrow \Gamma_0$ **has finite kernel and its image has finite index,** where Γ is a mapping class group.

The period map

REMARK: To simplify the language, we redefine Teich and Comp for hyperkähler manifolds, admitting only complex structures of Kähler type. Since the Hodge numbers are constant in families of Kähler manifolds, **for any $J \in$ Teich, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.**

Definition: Let Per : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map Per : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: Per **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}\text{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}\text{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$**.** Indeed, the group $SO(H^2(M, \mathbb{R}), q) = SO(b_2 - 3, 3)$ acts transitively on $\mathbb{P}$er, and $SO(2) \times SO(b_2 - 3, 1)$ is a stabilizer of a point.

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts, 2001) Two points $I, I' \in \operatorname{Teich}$ **are non-separable if and only if there exists a bimeromorphism $(M, I) \longrightarrow (M, I')$ which is non-singular in codimension 2 and acts as identity on $H^2(M)$.**

REMARK: This is possible only if (M, I) and (M, I') contain a rational curve. **General hyperkähler manifold has no curves;** ones which have belong to a countable union of divisors in Teich.

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism, for each connected component of Teich_b.

REMARK: The action of a lattice subgroup $\Gamma \subset O(H^2(M, \mathbb{Z}))$ on $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ **is ergodic (Moore),** and its orbits **are classified using Ratner's theorem.** This is the main tool in the arguments used today.