Zoll manifolds, conifold transform and symplectic packing

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Contact manifolds.

Definition: Let M be a smooth manifold, $\dim M = 2n-1$, and ω a symplectic form on $M \times \mathbb{R}^{>0}$. Suppose that ω is **automorphic**: $\Psi_q^* \omega = q^2 \omega$, where $\Psi_q(m,t) = (m, qt)$. Then M is called **contact**.

DEFINITION: The contact form on M is defined as $\theta = i_v \omega$, where $v = t\frac{d}{dt}$. Then $d\theta = \{d, i_v\}\omega = \operatorname{Lie}_v \omega = \omega$. Therefore, **the form $(d\theta)^{n-1} \wedge \theta = \frac{1}{n} \operatorname{Lie}_v \omega^n$ is non-degenerate on $M \times \{t_0\} \subset M \times \mathbb{R}^{>0}$.**

Remark: Usually, a contact manifold is defined as a **$(2n-1)$-manifold with 1-form θ such that $d\theta^{n-1} \wedge \theta$ is nowhere degenerate.**

Example: An odd-dimensional sphere S^{2n-1} is contact. Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{R}^{2n} \backslash 0$ has the standard symplectic form $\sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$ which is obviously homogeneous.

Contact geometry is an odd-dimensional counterpart to symplectic geometry

DEFINITION: The Reeb field on a contact manifold (M, θ) is a field $R \in TM$ such that $d\theta(R, \cdot) = 0$ and $\langle \theta, R \rangle = 1$.

Kähler manifolds.

Definition: Let (M, I) be a complex manifold, $\dim_{\mathbb{C}} M = n$, and g is Riemannian form. Then g is called **Hermitian** if $g(Ix, Iy) = g(x, y)$.

Remark: Since $I^2 = -\operatorname{Id}$, it is equivalent to $g(Ix, y) = -g(x, Iy)$. **The form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.**

Definition: The differential form ω is called **the Hermitian form of (M, I, g)**.

Definition: A complex Hermitian manifold is called **Kähler** if $d\omega = 0$.

Sasakian manifolds.

Definition: Let (M, g_M) be a Riemannian manifold, $\dim M = 2n-1$, and (g, ω, I) a Kaehler structure on $M \times \mathbb{R}^{>0}$ with $g = g_M + t^2 dt \otimes dt$. Suppose that ω is **automorphic**: $\Psi_q^* \omega = q^2 g$, where $\Psi_q(m,t) = (m, qt)$, and I is Ψ_q-invariant. Then M is called **Sasakian**, and $M \times \mathbb{R}^{>0}$ its **Kähler cone**.

Sasakian geometry is an odd-dimensional counterpart to Kähler geometry

Remark: A Sasakian manifold is obviosly contact. Indeed, **a Sasakian manifold is a contact manifold equipped with a compatible Riemannian metric.**

Example: An odd-dimensional sphere S^{2n-1} is Sasakian. Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{C}^n \backslash 0$ has the standard Kähler form $\sqrt{-1}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$ which is obviously automorphic.

S. Sasaki, "On differentiable manifolds with certain structures which are closely related to almost contact structure", Tohoku Math. J. 2 (1960), 459-476.

Boothby-Wang theorem

W. M. Boothby, H. C. Wang, On Contact Manifolds Annals of Mathematics, Second Series, Vol. 68, No. 3 (Nov., 1958), pp. 721-734.

DEFINITION: A contact manifold (M,θ) is **normal** if it is equipped with an S^1-action preserving θ and tangent to the Reeb field.

REMARK: Let (M,θ) be a contact manifold. **Then the form $d\theta$ is non-degenerate the bundle $\ker\theta \subset TM$.**

THEOREM: (Boothby-Wang, 1958)
Let (M,θ) be a normal contact manifold. **Then its space X of Reeb orbits is symplectic and the natural projection $\pi: M \longrightarrow X$ induces a symplectic isomorphism $d\pi: \ker\theta|_x \longrightarrow T_{\pi(x)}X$.**

THEOREM: (Boothby-Wang, 1958)
Let (M,θ) be a normal contact manifold, and (X,ω) the symplectic manifold obtained as its space of Reeb orbits. **Then the cohomology class of ω is integral.** Conversely, for any symplectic manifold (X,ω) with $[\omega] \in H^2(X,\mathbb{Z})$, **there exists a principal S^1-bundle L with $c_1(L) = [\omega]$ and a normal contact structure on $M := \mathrm{Tot}(L)$ such that the corresponding Boothby-Wang projection coincides with the natural map $M \longrightarrow X$.**

Boothby-Wang theorem for Sasakian manifolds

REMARK: Suppose that M is a Sasakian manifold with all Reeb orbits closed. Then its space X or Reeb orbits is a projective orbifold, and the Kähler cone $C(M)$ is the affine cone over $X \subset \mathbb{C}P^n$. The converse is also true: **whenever X is Kähler, the corresponding Boothby-Wang contact manifold is Sasakian.**

Hypersurfaces of contact type

DEFINITION: A vector field v on a symplectic manifold (M, ω) is called **Liouville** if $\operatorname{Lie}_v \omega = \omega$. We say that a smooth hypersurface $S \subset M$ is **a hypersurface of contact type** if there exists a Liouville vector field $v \in TM$ defined in a neighbourhood $U \supset S$ and transversal to S.

CLAIM: A hypersurface of contact type is contact, with the contact form given by $\alpha := i_v \omega|_S$.

Proof. Step 1: Since v is transversal to S, its orbit space is S. This can be used to identify its tubular neighbourhood U with $S \times I$, where I is an interval, in such a way that the shift in I multiplies the symplectic form ω by a scalar.

Step 2: We write $\omega = t dt \wedge \alpha + t^2 \omega_0$, where ω_0 is a 2-form on S and t the coordinate on I. In these notations, $v = t\frac{d}{dt}$. Then $t^2 \alpha = i_v \omega$. Therefore, $d\omega = 0$ implies that $d\alpha = -\omega_0$, hence $i_v \omega^n = \alpha \wedge (\omega_0)^n = \alpha \wedge (d\alpha)^n$ is non-degenerate on S. ■

REMARK: Gray stability theorem claims that **a smooth deformation of a compact contact manifold S is contactomorphic to S.** The space of Liouville vector fields in a neighbourhood of $S \subset M$ is convex, hence contractible. This implies that **the contact structure on a hypersurface of contact type is unique** up to a contact diffeomorphism.

Reeb field and almost complex structures

DEFINITION: An almost complex structure I is **compatible with a symplectic structure** ω if $\omega(Ix, Iy) = \omega(x, y)$ and $\omega(x, Ix) > 0$ for any $x \neq 0$. **In this case, $g(x, y) := \omega(x, Iy)$ is a positive definite scalar product.**

PROPOSITION 1: Let S be a contact manifold and $(C(S), \omega)$ its symplectic cone, equipped with the symplectic homothety diffeomorphism Ψ_t. Consider an Ψ_t-invariant almost complex structure I on $C(S)$. Assume that the vector field d/dt satisfies $|d/dt| = 1$ and is orthogonal to $S \subset C(S)$ embedded as $S \times \{r\}$. **Then the Reeb field can be expressed as $R = I(d/dt)$,** where t is the coordinate on $\mathbb{R}^{>0}$, considered as a function on $C(S) = S \times \mathbb{R}^{>0}$.

Proof: For any $x \in TC(S)$, we have $x \in TS$ if and only if $x \perp \frac{d}{dt}$. Therefore, the symplectic orthogonal to TS is $I(d/dt)$; this vector field clearly has constant length. ■

Geodesic flow on a Riemannian manifold as a Hamiltonian flow

REMARK: Recall that **a Hamiltonian vector field** on a symplectic manifold M is a vector field v which is symplectically dual to dH, where H is a smooth function, called **the Hamiltonian** of v.

DEFINITION: Let M be a complete Riemannian manifold, $(m, v) \in TM$ a point in its tangent space, and $\gamma_{(m,v)}(t)$ the geodesic starting in m and tangent to v. **The geodesic flow** is a diffeomorphism flow $\Psi_t, t \in \mathbb{R}$ on the tangent bundle TM taking $(m, v) \in TM$ to $(\gamma_{(m,v)}(t), \dot{\gamma}_{(m,v)}(t)) \in T_{\gamma_{(m,v)}(t)}M$.

CLAIM: Let M be a Riemannian manifold. We use the Riemannian metric to identify TM and T^*M. This identification gives a symplectic structure on TM. Denote by H the function $H(v) = |v|^2$. **Then the geodesic flow on TM is the Hamiltonian flow associated with the function H.**

Proof: See *V. I. Arnold, Mathematical Methods of Classical Mechanics.* ■

Contact manifold associated with the cotangent bundle

CLAIM: Let M be a smooth manifold, and ω the Hamilton symplectic form on T^*M. Then **the set $ST^*M := \{v \in T^*M \quad | \quad |v| = 1\}$ is a hypersurface of contact type.**

Proof: In coordinates the form ω can be written as $\sum dp_i \wedge dq_i$, where p_i are coordinates on M and q_i the corresponding coordinates on the fibers of the bundle $T^*M \longrightarrow M$. The fiberwise homothety vector field $\sum q_i d/dq_i$ is a Liouville field, transversal to ST^*M. ■

Reeb orbits and the geodesic flow

PROPOSITION: In the above assumptions, the Reeb field of ST^*M is equal to the symplectic dual of dH, that is, **it is the vector field generating the geodesic flow.**

Proof. Step 1: To use Proposition 1, we need to construct a compatible almost complex structure which is invariant with respect to the homothety map. Let $\pi : T^*M \longrightarrow M$ be the projection. Using the Levi-Civita connection, we obtain a decomposition $TT^*M = \pi^*T^*M \oplus \pi^*TM$. The symplectic structure on T^*M is induced by the natural pairing of these two factors.

The metric on M induces a Riemannian metric on TT^*M, called **the Sasaki metric**. The corresponding almost complex structure uses the decomposition $TT^*M = \pi^*T^*M \oplus \pi^*TM$, with the first term $T_{vert}T^*M$ consisting of fiberwise tangent vector fields, and the second term $T_{hor}T^*M$ the "horizontal sub-bundle", obtained using the connection. The almost complex structure exchanges $T_{vert}T^*M$ identified with π^*TM using the metric and $T_{hor}T^*M = \pi^*TM$. Under this identification the radial vector field becomes the vector field which is horizontal and equal to v in $(m, v) \in TM = T^*M$; this is precisely the vector field tangent to the geodesic flow.

Step 2: By Proposition 1, the Reeb field is $F(v)$, where v is the radial vector field tangent to the homothety. ■

Zoll manifolds and contact manifolds

DEFINITION: A compact Riemannian manifold M is called **Zoll** if all its geodesics are compact and the geodesics have constant length

PROPOSITION: Let M be a compact Riemannian manifold, and ST^*M the manifold of unit cotangent vectors, considered as a contact manifold. **Then ST^*M is a normal contact manifold if and only if M is Zoll.**

Proof. Step 1: Let Z be a Riemannian manifold equipped with a rank 1 foliation $\mathcal{F}$ with compact fibers, and $Z \longrightarrow Z/\mathcal{F}$ be the projection to the leaf space. **Then $Z/\mathcal{F}$ is smooth if and only if length of an orbit is a continuous function on $Z/\mathcal{F}$.** This is left as an exercise.

Step 2: We apply this observation to the geodesic flow on ST^*M. If M is Zoll, this implies that the the projection from ST^*M to the space of Reeb orbits is smooth, hence ST^*M is normal; converse follows from Wadsley theorem. ■

Zoll manifolds and Kähler geometry

EXAMPLE: In dimension 2, there are many non-trivial metrics on 2-spheres ("Zoll spheres"), and a symmetric metric on $\mathbb{R}P^2$. In bigger dimension, the only known Zoll manifolds are S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, and the Cayley projective plane $\mathbb{C}aP^2$.

PROPOSITION: Let M be S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, or $\mathbb{C}aP^2$. **Then the contact manifold ST^*M is Sasakian**, and its space of Reeb orbits is Kähler.

REMARK: For S^n the space of Reeb orbits is $\mathrm{Gr}_2(\mathbb{R}^{n+1})$, with its natural structure of the Kähler symmetric space. For $\mathbb{C}P^n$ it is the space of $1,2$-flags (point and a line) in $\mathbb{C}P^n$. For $\mathbb{H}P^n$ and $\mathbb{C}aP^2$ we don't know.

The conifold transform for Calabi-Yau threefolds

Smith, I.; Thomas, R. P.; Yau, S.-T., Symplectic conifold transitions, J. Differential Geom. 62 (2002), no. 2, 209-242.

DEFINITION: **A Calabi-Yau threefold** is a complex 3-dimensional compact Kähler manifold with $c_1(M) = 0$.

REMARK: Let $S \subset M$ be a smooth $\mathbb{C}P^1$ on a Calabi-Yau manifold. In the typical situation, $NS \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. By Grauert theorem, such a rational curve can be blown down, defining a map $M \longrightarrow M_0$, where M_0 is a singular complex variety. It is not hard to see that **this singularity is locally biholomorphic to an affine cone over $\mathbb{C}P^1 \times \mathbb{C}P^1$.**

PROPOSITION: Let $S \subset M$ be a smooth rational curve on a Calabi-Yau manifold, with $NS \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and M_0 the singular variety obtained by a blow-down of S. **Then M_0 admits a smooth deformation M_1.**

DEFINITION: The transition from M to M_1 is called **the conifold transition.**

Miles Reed's conjecture

REMARK: The manifold M_1 has trivial canonical bundle. If it satisfies $b_2(M_1) = 0$, this manifold is diffeomorphic to $\#_i S^3 \times S^3$ (connected sum of several copies of $S^3 \times S^3$). Miles Reed **conjectured that any Calabi-Yau can be devolved to such a manifold after a sequence of conifold transitions.**

Note that the conifold transform in complex-analytic setup **is emphatically non-invertible: there exists a Calabi-Yau manifold with a Lagrangian S^3 which cannot be blown down.**

Smith-Thomas and Yau described the topology of conifold transition symplectically.

Gluing symplectic manifolds over contact hypersurfaces

DEFINITION: Let $S \subset M$ be a contact-type boundary hypersurface in a symplectic manifold. We say that S is **convex** if the Liouville field is directed from M to S, and **concave** otherwise.

PROPOSITION: Let $S_1 \subset M_1$ be a concave component of a boundary of a symplectic manifold, and $S_2 \subset M_2$ a convex component. Assume that S_1 is isomorphic to S_2 as contact manifold. **Then one can glue M_1 to M_2 by taking an appropriate contact diffeomorphism, identifying S_1 with S_2.**

Proof: Since M_i is symplectomorphic to a symplectic cone in a neighbourhood of S_i, it would suffice to pick a contactomorphism $S_1 \longrightarrow S_2$ which can be extended to a cone. However, any contactomorphism can be extended to a cone, by construction. ■

Symplectic manifolds with normal contact boundary

EXAMPLE: Let M be a symplectic threefold, and $S^3 \subset M$ a Lagrangian 3-sphere. By Weinstein neighbourhood theorem, there is a neighbourhood of S in M which is symplectomorphic to a manifold of open balls in T^*S^3 (for sufficiently small radius of the ball). **Its boundary Z is a Boothby-Wang contact manifold, which is S^1-fibered over the space of geodesics in S^3,** identified with $\mathbb{C}P^1 \times \mathbb{C}P^1$ Then Z is a boundary of the cone over $\mathbb{C}P^1 \times \mathbb{C}P^1$ associated with the ample bundle $\mathcal{O}(1,1)$.

We glue the corresponding singular complex variety in Z, using the gluing theorem. This replaces the Lagrangian S^3 with a neighbourhood of zero in the affine cone over $\mathbb{C}P^1 \times \mathbb{C}P^1$.

CLAIM: A small resolution of this conical singularity **is biholomorphic to a neighbourhood of a rational curve C** in a Calabi-Yau manifold, with $NC = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

REMARK: Unlike the construction with complex deformations, **this construction is invertible**: by Weinstein neighbourhood theorem, **any symplectic submanifold in a symplectic manifold has a neighbourhood symplectomorphic to its neighbourhood in the normal bundle.**

The conifold transition, a general definition

DEFINITION: Let M be a symplectic manifold, $L \subset M$ a Lagrangian submanifold admitting a Zoll metric. Consider a Weinstein neighbourhood U_L with normal (in Boothby-Wang sense) contact boundary. Denote by Z_L the corresponding singular complex variety, $\partial Z_L = \partial U_L$, and let V_L be a complex resolution of Z_L. **The direct conifold transform** is obtained by replacing U_L with V_L; it replaces a Lagrangian submanifold by a symplectic submanifold.

DEFINITION: Its inverse is defined in a similar way: given a smooth symplectic submanifold $X \subset M$, isomorphic to the preimage of the singularity in the resolution map $V_L \longrightarrow Z_L$, with normal bundle isomorphic to the normal bundle of $X \subset V_L$, **the inverse conifold transform uses the Weinstein neighbourhood theorem to replace V_L with U_L.**

EXAMPLE: The symplectic conifold transition of Smith-Thomas and Yau is an example of this construction.

The conifold transition, examples and applications

EXAMPLE: For dimension $n \geqslant 4$ the cone over $\mathsf{Gr}_2(\mathbb{R}^n)$ does not have partial resoltions, this means that **we can only replace a Lagrangian sphere S^n with a Grassmannian $\mathsf{Gr}_2(\mathbb{R}^n)$, symplectically embedded to an ambient manifoldm, and vice versa.**

EXAMPLE: The cone over (1,2)-flags (the space of geodesics in $\mathbb{C}P^n$) **has a partial resolution which is bimeromorphic to a holomorphic cotangent bundle to $\mathbb{C}P^n$.** The corresponding conifold transform replaces a Lagrangian $\mathbb{C}P^n$ by a symplectic $\mathbb{C}P^n$ and vice versa, similar to the hyperkähler rotation.

This construction is useful for symplectic packing: if we have a control over the volume of V_L, we obtain control over volume of U_L and vice versa.

This brings the following result about K3 surfaces.

THEOREM: Let $L_1, ..., L_n$ be a collection of non-intersecting special Lagrangian 2-spheres in a K3 surface M, and $u_1, ..., u_n$ a numbers which satisfy $\sum u_i \leqslant \mathsf{Vol}_\omega M$. Denote by U_{L_i} the Weinstein neighbourhood of S^2 in T^*S^2 which is invariant under rotations and has symplectic volume u_i; such a neighbourhood is clearly unique. **Then there exists a collection of symplectic embeddings $\varphi_i: \ U_{L_i} \longrightarrow M$, with non-intersecting images, taking the zero section $S^2 \subset U_{L_i} \subset T^*S^2$ to $L_i \subset M$.**