Conifold transform and symplectic structures on orbifold K3 surfaces

Misha Verbitsky

Estruturas geométricas em variedades,

IMPA, December 21, 2023,

Joint work with Michael Entov

Weinstein normal neighbourhood theorems

THEOREM: (Weinstein Lagrangian neighbourhood theorem)
Let $X \subset M$ be a compact Lagrangian submanifold in (M, ω). Then **there exists a neighbourhood U of $X \subset M$ which is symplectomorphic to a neighbourhood of X in $X \subset T^*X$.**

If $X \subset M$ is a symplectic submanifold, the normal bundle NX is equipped with a natural symplectic structure. This is used to state another normal neighbourhood theorem

THEOREM: (Weinstein symplectic neighbourhood theorem)
Let M_1, M_2 be symplectic manifolda, $X_1 \subset M_1$, $X_2 \subset M_2$ be symplectic submanifolds. Consider the normal bundles NX_i with the induced symplectic structure. Assume that there exists a symplectomorphism $\nu : X_1 \longrightarrow X_2$ such that the symplectic vector bundle $\nu^* NX_2$ is isomorphic to NX_1. Then **a neighbourhood of X_1 is symplectomorphic to a neighbourhood of X_2.**

Both results are due to A. Weinstein, *Alan Weinstein, Symplectic manifolds and their lagrangian submanifolds, Advances in Mathematics, Vol. 6 (3), June 1971, pp. 329-346*

Contact manifolds.

Definition: Let M be a smooth manifold, $\dim M = 2n-1$, and ω a symplectic form on $M \times \mathbb{R}^{>0}$. Suppose that ω is **automorphic**: $\Psi_q^*\omega = q^2\omega$, where $\Psi_q(m,t) = (m,qt)$. Then M is called **contact**.

DEFINITION: The contact form on M is defined as $\theta = i_v\omega$, where $v = t\frac{d}{dt}$. Then $d\theta = \{d, i_v\}\omega = \operatorname{Lie}_v \omega = \omega$. Therefore, **the form $(d\theta)^{n-1} \wedge \theta = \frac{1}{n} \operatorname{Lie}_v \omega^n$ is non-degenerate on $M \times \{t_0\} \subset M \times \mathbb{R}^{>0}$.**

Remark: Usually, a contact manifold is defined as a **$(2n-1)$-manifold with 1-form θ such that $d\theta^{n-1} \wedge \theta$ is nowhere degenerate.**

Example: An odd-dimensional sphere S^{2n-1} is contact. Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{R}^{2n} \backslash 0$ has the standard symplectic form $\sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$ which is obviously homogeneous.

Contact geometry is an odd-dimensional counterpart to symplectic geometry

DEFINITION: The Reeb field on a contact manifold (M,θ) is a field $R \in TM$ such that $d\theta(R, \cdot) = 0$ and $\langle \theta, R \rangle = 1$.

Boothby-Wang theorem

W. M. Boothby, H. C. Wang, On Contact Manifolds Annals of Mathematics, Second Series, Vol. 68, No. 3 (Nov., 1958), pp. 721-734.

DEFINITION: A contact manifold (M, θ) is **normal** if it is equipped with an S^1-action preserving θ and tangent to the Reeb field.

REMARK: Let (M, θ) be a contact manifold. **Then the form $d\theta$ is non-degenerate the bundle $\ker\theta \subset TM$.**

THEOREM: (Boothby-Wang, 1958)
Let (M, θ) be a normal contact manifold. **Then its space X of Reeb orbits is symplectic and the natural projection $\pi : M \longrightarrow X$ induces a symplectic isomorphism $d\pi : \ker\theta|_x \longrightarrow T_{\pi(x)}X$.**

THEOREM: (Boothby-Wang, 1958)
Let (M, θ) be a normal contact manifold, and (X, ω) the symplectic manifold obtained as its space of Reeb orbits. **Then the cohomology class of ω is integral.** Conversely, for any symplectic manifold (X, ω) with $[\omega] \in H^2(X, \mathbb{Z})$, **there exists a principal S^1-bundle L with $c_1(L) = [\omega]$ and a normal contact structure on $M := \operatorname{Tot}(L)$ such that the corresponding Boothby-Wang projection coincides with the natural map $M \longrightarrow X$.**

Reeb field and almost complex structures

DEFINITION: An almost complex structure I is **compatible with a symplectic structure** ω if $\omega(Ix, Iy) = \omega(x, y)$ and $\omega(x, Ix) > 0$ for any $x \neq 0$. **In this case, $g(x, y) := \omega(x, Iy)$ is a positive definite scalar product.**

PROPOSITION 1: Let S be a contact manifold and $(C(S), \omega)$ its symplectic cone, equipped with the symplectic homothety diffeomorphism Ψ_t. Consider an Ψ_t-invariant almost complex structure I on $C(S)$. Assume that the vector field d/dt satisfies $|d/dt| = 1$ and is orthogonal to $S \subset C(S)$ embedded as $S \times \{r\}$. **Then the Reeb field can be expressed as $R = I(d/dt)$,** where t is the coordinate on $\mathbb{R}^{>0}$, considered as a function on $C(S) = S \times \mathbb{R}^{>0}$.

Proof: For any $x \in TC(S)$, we have $x \in TS$ if and only if $x \perp \frac{d}{dt}$. Therefore, the symplectic orthogonal to TS is $I(d/dt)$; this vector field clearly has constant length. ∎

Geodesic flow on a Riemannian manifold as a Hamiltonian flow

REMARK: Recall that **a Hamiltonian vector field** on a symplectic manifold M is a vector field v which is symplectically dual to dH, where H is a smooth function, called **the Hamiltonian** of v.

DEFINITION: Let M be a complete Riemannian manifold, $(m, v) \in TM$ a point in its tangent space, and $\gamma_{(m,v)}(t)$ the geodesic starting in m and tangent to v. **The geodesic flow** is a diffeomorphism flow $\Psi_t, t \in \mathbb{R}$ on the tangent bundle TM taking $(m, v) \in TM$ to $(\gamma_{(m,v)}(t), \dot{\gamma}_{(m,v)}(t)) \in T_{\gamma_{(m,v)}(t)}M$.

CLAIM: Let M be a Riemannian manifold. We use the Riemannian metric to identify TM and T^*M. This identification gives a symplectic structure on TM. Denote by H the function $H(v) = |v|^2$. **Then the geodesic flow on TM is the Hamiltonian flow associated with the function H.**

Proof: See *V. I. Arnold, Mathematical Methods of Classical Mechanics.* ■

Reeb orbits and the geodesic flow

PROPOSITION: In the above assumptions, the Reeb field of ST^*M is equal to the symplectic dual of dH, that is, **it is the vector field generating the geodesic flow.**

Proof. Step 1: To use Proposition 1, we need to construct a compatible almost complex structure which is invariant with respect to the homothety map. Let $\pi : T^*M \longrightarrow M$ be the projection. Using the Levi-Civita connection, we obtain a decomposition $TT^*M = \pi^*T^*M \oplus \pi^*TM$. The symplectic structure on T^*M is induced by the natural pairing of these two factors.

The metric on M induces a Riemannian metric on TT^*M, called **the Sasaki metric**. The corresponding almost complex structure uses the decomposition $TT^*M = \pi^*T^*M \oplus \pi^*TM$, with the first term $T_{vert}T^*M$ consisting of fiberwise tangent vector fields, and the second term $T_{hor}T^*M$ the "horizontal sub-bundle", obtained using the connection. The almost complex structure exchanges $T_{vert}T^*M$ identified with π^*TM using the metric and $T_{hor}T^*M = \pi^*TM$. Under this identification the radial vector field becomes the vector field which is horizontal and equal to v in $(m, v) \in TM = T^*M$; this is precisely the vector field tangent to the geodesic flow.

Step 2: By Proposition 1, the Reeb field is $I(v)$, where v is the radial vector field tangent to the homothety. ■

The Reeb field on S^3

Let S be S^3 with $SU(2)$-invariant contact structure. Then the Reeb field is also $SU(2)$-invariant, and its orbits are translations of 1-parametric subgroups. Since **all 1-parametric subgroups in $S(2)$ are closed, the Reeb foliation coincides with the Hopf foliation.**

COROLLARY: A left $SU(2)$-invariant contact structure on S^3 **is unique up to the right $SU(2)$-action**. It can be obtained as an orthogonal complement to the tangent space to Hopf foliation. ■

COROLLARY: Let $L \subset M$ be a Lagrangian 2-sphere in a symplectic 4-manifold. **Then L has a symplectic neighbourhood with a boundary which is contact equivalent to $\mathbb{R}P^3$.**

Proof: From the description above, it is clear that there exists a normal neighbourhood of L which admits an $U(2)$-action by symplectomorphisms, extending the $U(2)$-action on $\mathbb{C}P^1$. Since $T^*\mathbb{C}P^1 = \frac{\operatorname{Tot}(\mathcal{O}(-1))}{\pm 1}$, which is a blow-up of $\mathbb{C}^2$, the boundary of a unit ball in $T^*\mathbb{C}P^1$ is $S^3/\pm 1$. **An $U(2)$-invariant contact structure on $\mathbb{R}P^3$ is unique up to a $U(2)$-action,** as shown above. ■

Hypersurfaces of contact type

DEFINITION: A vector field v on a symplectic manifold (M,ω) is called **Liouville** if $\operatorname{Lie}_v \omega = \omega$. We say that a smooth hypersurface $S \subset M$ is **a hypersurface of contact type** if there exists a Liouville vector field $v \in TM$ defined in a neighbourhood $U \supset S$ and transversal to S.

CLAIM: A hypersurface of contact type is contact, with the contact form given by $\alpha := i_v\omega|_S$.

Proof. Step 1: Since v is transversal to S, its orbit space is S. This can be used to identify its tubular neighbourhood U with $S \times I$, where I is an interval, in such a way that the shift in I multiplies the symplectic form ω by a scalar.

Step 2: We write $\omega = tdt \wedge \alpha + t^2\omega_0$, where ω_0 is a 2-form on S and t the coordinate on I. In these notations, $v = t\frac{d}{dt}$. Then $t^2\alpha = i_v\omega$. Therefore, $d\omega = 0$ implies that $d\alpha = -\omega_0$, hence $i_v\omega^n = \alpha \wedge (\omega_0)^n = \alpha \wedge (d\alpha)^n$ is non-degenerate on S. ■

REMARK: Gray stability theorem claims that **a smooth deformation of a compact contact manifold S is contactomorphic to S.** The space of Liouville vector fields in a neighbourhood of $S \subset M$ is convex, hence contractible. This implies that **the contact structure on a hypersurface of contact type is unique** up to a contact diffeomorphism.

Contact manifold associated with the cotangent bundle

CLAIM: Let M be a smooth manifold, and ω the Hamilton symplectic form on T^*M. Then **the set $ST^*M := \{v \in T^*M \ \ | \ \ |v| = 1\}$ is a hypersurface of contact type.**

Proof: In coordinates the form ω can be written as $\sum dp_i \wedge dq_i$, where p_i are coordinates on M and q_i the corresponding coordinates on the fibers of the bundle $T^*M \longrightarrow M$. The fiberwise homothety vector field $\sum q_i d/dq_i$ is a Liouville field, transversal to ST^*M. ■

Gluing symplectic manifolds over contact hypersurfaces

DEFINITION: Let $S \subset M$ be a contact-type boundary hypersurface in a symplectic manifold. We say that S is **convex** if the Liouville field is directed from M to S, and **concave** otherwise.

PROPOSITION: Let $S_1 \subset M_1$ be a concave component of a boundary of a symplectic manifold, and $S_2 \subset M_2$ a convex component. Assume that S_1 is isomorphic to S_2 as contact manifold. **Then one can glue M_1 to M_2 by taking an appropriate contact diffeomorphism, identifying S_1 with S_2.**

Proof: Since M_i is symplectomorphic to a symplectic cone in a neighbourhood of S_i, **it would suffice to pick a contactomorphism $S_1 \longrightarrow S_2$ which can be extended to a cone**. However, **any contactomorphism can be extended to a cone,** by definition. ■

Example: the conifold transform on K3 surfaces

Consider the complex manifold $T^*\mathbb{C}P^1$, and let $\varphi(v) := |v|^2$. It is not hard to see that $dId\varphi$ is a Kähler metric outside of the zero section, and the level set S of φ is contact. Since $T^*\mathbb{C}P^1 = \frac{\operatorname{Tot}(\mathcal{O}(-1))}{\pm 1}$, which is a blow-up of $\mathbb{C}^2$, we obtain that **S is a contact manifold which is isomorphic to $\mathbb{R}P^3$ with the standard contact structure.**

COROLLARY: By Weinstein symplectic neighbourhood theorem, **any symplectic S^2 in a symplectic 4-manifold M with $c_1(M) = 0$ admits a neighbourhood with a neighbourhood with the contact boundary isomorphic to $\mathbb{R}P^3$.**

DEFINITION: Let S be a Lagrangian sphere in a K3 surface. Using the previous arguments, we can replace a neighbourhood U of S by a neighbourhood of symplectic S^2 and glue it in place of U. This construction is an example of a **conifold transform**, defined below in full generality.

Zoll manifolds and contact manifolds

DEFINITION: A compact Riemannian manifold M is called **Zoll** if all its geodesics are compact and the geodesics have constant length

PROPOSITION: Let M be a compact Riemannian manifold, and ST^*M the manifold of unit cotangent vectors, considered as a contact manifold. **Then ST^*M is a normal contact manifold if and only if M is Zoll.**

Proof. Step 1: Let Z be a Riemannian manifold equipped with a rank 1 foliation $\mathcal{F}$ with compact fibers, and $Z \longrightarrow Z/\mathcal{F}$ be the projection to the leaf space. **Then $Z/\mathcal{F}$ is smooth if and only if length of an orbit is a continuous function on $Z/\mathcal{F}$.** This is left as an exercise.

Step 2: We apply this observation to the geodesic flow on ST^*M. If M is Zoll, this implies that the the projection from ST^*M to the space of Reeb orbits is smooth, hence ST^*M is normal; converse follows from Wadsley theorem. ■

The conifold transition, a general definition

DEFINITION: Let M be a symplectic manifold, $L \subset M$ a Lagrangian submanifold admitting a Zoll metric. Consider a Weinstein neighbourhood U_L with normal (in Boothby-Wang sense) contact boundary. Denote by Z_L the corresponding singular complex variety, $\partial Z_L = \partial U_L$, and let V_L be a complex (or symplectic) partial resolution of singularities for the cone Z_L. **The direct conifold transform** is obtained by replacing U_L with V_L; it replaces a Lagrangian submanifold by a symplectic submanifold.

DEFINITION: Its inverse is defined in a similar way: given a smooth symplectic submanifold $X \subset M$, isomorphic to the preimage of the singularity in the resolution map $V_L \longrightarrow Z_L$, with normal bundle isomorphic to the normal bundle of $X \subset V_L$, **the inverse conifold transform uses the Weinstein neighbourhood theorem to replace V_L with U_L.**

Teichmüller space for symplectic structures

For a K3 surface, this construction replaces a Lagrangian sphere by a symplectic sphere, in a way which is almost (but not quite) functorial. To make sense of it functorially, we define the symplectic Teichmüller spaces.

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. **Then $\Gamma(\Lambda^2 M)$ is a Fréchet vector space, and Symp a Fréchet manifold.**

DEFINITION: Consider the group of diffeomorphisms, denoted Diff, as a Fréchet Lie group, and denote its connected component ("group of isotopies") by Diff_0. The quotient group $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ is called **the mapping class group** of M.

DEFINITION: Teichmüller space of symplectic structures on M is defined as a quotient $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$.

Moser theorem

DEFINITION: Two symplectic structures are called **isotopic** if they lie in the same orbit of Diff_0.

DEFINITION: Define **the period map** $\mathrm{Per}: \mathrm{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\mathrm{Per}: \mathrm{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is locally a diffeomorphism.**

Teichmüller space for Lagrangian submanifolds

DEFINITION: Fix a symplectic manifold (M,ω), and let $L \subset M$ be a Lagrangian submanifold with. Denote by Symp_L an infinite-dimensional space of symplectic forms ω on M such that $\omega|_L = 0$. We define **the Teichmüller space of pairs (ω, L)** as the quotient $\operatorname{Symp}_L / \operatorname{Diff}^0_L$, where Diff^0_L is the group of diffeomorphisms of M which preserve $L \subset M$, and Diff^0_L its connected component.

REMARK: The space $\operatorname{Symp}_L / \operatorname{Diff}^0_L$ is always smooth, but it is harder to prove this when $b_1(L) > 0$, because the period map is more difficult to define. Therefore, we restrict ourselves with the case $b_1(L) = 0$.

Moser theorem for the Teichmüller space of Lagrangian subvarieties

THEOREM: Let $W \subset H^2(M,\mathbb{R})$ be the space of all cohomology classes on M which vanish on L, and $\operatorname{Per}_L: \operatorname{Symp}_L / \operatorname{Diff}^0_L \longrightarrow W$ take (M,ω) to the cohomology class of ω. **Then** Per **is locally a diffeomorphism**.

Proof. Step 1: It would suffice to prove that a small deformation ω_1 of a symplectic form ω which is Lagrangian on L is isotopic to a form which is also Lagrangian on L, assuming that ω_1 is exact on L. **This is equivalent to constructing a Lagrangian submanifold L_1 of (M,ω_1) which can be isotopically mapped to L, and proving it is unique up to isotopy.**

Step 2: Take a Weinstein neighbourhood U_L of L in (M,ω). Then $\omega_1 = \omega - d\theta$, where θ is a 1-form. Denote by L_1 the graph of θ in T^*L. For ω_1 sufficiently close to ω and θ sufficiently small, we may assume that L_1 belongs to U_L. On a graph of a map $\theta: L \longrightarrow T^*L$, the form ω is equal to $d\theta$, hence $L_1 \subset U_L$ is Lagrangian. **This shows that the period map $\operatorname{Per}_L: \operatorname{Symp}_L / \operatorname{Diff}^0_L \longrightarrow W$ is locally a surjection.**

Step 3: The period map is locally injective, because $b_1(L) = 0$, hence any closed 1-form is exact, **which implies that all Lagrangian sections of $T^*L \longrightarrow L$ are Hamiltonian isotopic.** ■

Orbifolds

REMARK: Unlike Lagrangian spheres, **the symplectic spheres have a numeric invariant:** the symplectic volume. If we obtain a symplectic sphere as a result of the conifold transform, its volume depends on the choice of the Weinstein neighbourhood U_L, which is arbitrary for small volumes, but not for big volumes. **It is more convenient to blow down the sphere, and consider an orbifold.**

DEFINITION: **A smooth orbifold** is a topological space equipped with a covering by open sets of form B_i/Γ_i, where B_i is an open ball, and Γ_i a finite group acting on B_i, in such a way that the transition maps phi_{ij} : $B_i/\Gamma_i \longrightarrow B_j/\Gamma_j$ are naturally lifted to smooth maps $\tilde{\varphi}_{ij}: B_i \longrightarrow B_j$.

DEFINITION: **A symplectic (complex, Kähler, etc) orbifold** is an orbifold $(M, \{B_i/\Gamma_i\})$ such that every ball B_i is symplectic (complex, Kähler, etc), the action of Γ_i preserves the symplectic (complex, Kähler, etc) structure, and the transition maps $\tilde{\varphi}_{ij}: B_i \longrightarrow B_j$ are compatible with the symplectic (complex, Kähler, etc) structure.

Orbifold K3 surfaces

EXAMPLE: (symplectic blowdown on a K3 surface)
Consider a symplectic 2-sphere S in a K3 surface. Using Weinstein neighbourhood theorem, we may assume that S has a neighbourhood V_S isomorphic to a neighbourhood of S in T^*S. The boundary of V_S is contactomorphic to $\mathbb{R}P^3$ with the standard contact structure. Using the conifold transform, we replace V_S by $B/\pm 1$, where B is a ball in $\mathbb{C}^2$ with the standard symplectic structure. This gives an orbifold, called **an orbifold K3 surface**.

DEFINITION: This correspondence (or the same operation applied to a collection of non-intersecting Lagrangian subvarieties) is called **the orbifold conifold transform**.

DEFINITION: More generally, consider a symplectic or complex orbifold M with all singularities of form B/Γ, where $\Gamma \subset SU(2)$ is a finite subgroup. Assume that after the symplectic or complex blow-up, M is diffeomorphic to a K3 surface. Then M is called **an orbifold symplectic or complex K3 surface**.

REMARK: Let Γ be a finite group acting on a complex K3 surface M by holomorphic symplectic automorphisms. It is possible to show that M/Γ **is an orbifold K3 surface.** However, **not all orbifold K3 surfaces are obtained this way**.

Conifold transition for K3 surfaces

The main result today's talk

THEOREM: Let $L_1, ..., L_n$ be a collection of Lagrangian 2-spheres in a K3 surface M, and $\operatorname{Teich}_{L_1,...L_n}$ the corresponding symplectic Teichmüller space. Consider an orbifold K3 surface M_1 obtained from the orbifold conifold transform, and let $\operatorname{Teich}_{M_1}$ be its symplectic Teichmüller space. **The orbifold conifold transform defines a diffeomorphism** $\operatorname{Teich}_{L_1,...L_n} \longrightarrow \operatorname{Teich}_{M_1}$**.**

Proof: It would suffice to show that the result of direct and inverse transform is independent from the choice of Weinstein neighbourhood, but the group of Hamiltonian symplectomorphisms acs on the set of sufficiently small Weinstein neighbourhoods of given volume transitively. ■

Donaldson's conjecture is false for orbifold K3 surfaces

CONJECTURE: (Donaldson)
All symplectic structures on a K3 surface are compatible with a Kähler structure.

This conjecture is still open. However, an orbifold version of this conjecture is false, which is implied by the conifold transform.

THEOREM: There exists an orbifold symplectic K3 surface M with a single double point **not admitting an orbifold Kähler structure**.

Proof. Step 1: As follows from a result of Amerik-V. (2015), the Teichmüller space of symplectic structures compatible with a Kähler structure is Hausdorff and connected (the same argument works for Kähler K3 orbifolds). Suppose, on contrary, that any orbifold symplectic K3 surface with a single double point admits a Kähler structure. This would imply that the Teichmüller space Teich_L is connected, **and, indeed, that in each homotopy class of 2-spheres on a K3 surface there exists at most one (up to a Lagrangian isotopy) Lagrangian sphere.**

Step 2: This is impossible, by a result of P. Seidel (2000): **for any Lagrangian 2-sphere S in a symplectic K3 surface, there exists infinitely many Lagrangian spheres in the same smooth isotopy class,** which are not Lagrangian isotopic. ■