Lagrangian blow-up and blow-down for 4-dimensional symplectic manifolds

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Symplectic quotient

DEFINITION: Let ρ be an S^1-action on a symplectic manifold (M,ω) preserving the symplectic structure, and $\vec{v}$ its unit tangent vector. Cartan's formula gives $0 = \mathrm{Lie}_{\vec{v}}\omega = d(\omega \lrcorner \vec{v})$, hence $\omega \lrcorner \vec{v}$ is a closed 1-form. **Hamiltonian**, or **moment map** of ρ is an S^1-invariant function μ such that $d\mu = \omega \lrcorner \vec{v}$, and **symplectic quotient** $M /\!\!/_c S^1$ is $\mu^{-1}(c)/S^1$.

REMARK: In these assumptions, restriction of the symplectic form ω to $\mu^{-1}(c)$ vanishes on $\vec{v}$, hence it is **obtained as a pullback of a closed 2-form $\omega_{/\!\!/}$ on $M /\!\!/_c S^1$.**

THEOREM: The form $\omega_{/\!\!/}$ is a symplectic form on $M /\!\!/_c S^1$. In other words, **the symplectic quotient is a symplectic manifold.**

REMARK: If, in addition, M is equipped with a Kähler structure (I,ω), and S^1-action preserves the complex structure, the symplectic quotient $M /\!\!/_c S^1$ inherits the Kähler structure. In this case it is called **a Kähler quotient**. Whenever the S^1-action can be integrated to holomorphic $\mathbb{C}^*$-action, **the Kähler quotient is identified with an open subset of its orbit space.**

REMARK: The moment map is defined by $d\mu = \omega \lrcorner \vec{v}$ uniquely up to a constant. However, the symplectic quotient $M /\!\!/_c S^1 = \mu^{-1}(c)/S^1$ **depends heavily on the choice of $c \in \mathbb{R}$.**

Symplectic blow-up

CLAIM: Consider the standard S^1-action on $\mathbb{C}^n$, and let $W \subset \mathbb{C}^n$ be an S^1-invariant open subset. Consider the product $V := W \times \mathbb{C}$ with the standard symplectic structure and take the S^1-action on $\mathbb{C}$ opposite to the standard one. **Then its moment map is $w - t$, where $w(x) = |x|^2$ is the length function on W and $r(t) = |t|^2$ the length function on $\mathbb{C}$.**

DEFINITION: **Symplectic cut** of W is $(W \times \mathbb{C})/\!\!/_c S^1$.

REMARK: Geometrically, **the symplectic cut is obtained as follows.** Take $c \in \mathbb{R}$, and let $W_c := \{w \in W \ \ | \ \ |w|^2 \leqslant c\}$. Then W_c is a manifold with boundary ∂W_c, which is a sphere $|w|^2 = c$. Then $(W \times \mathbb{C})/\!\!/_c S^1 = (W_c \times \mathbb{C})/\!\!/_c S^1$ is obtained from W_c by gluing each S^1-orbit which lies on ∂W_c to a point. Combinatorially, $(W \times \mathbb{C})/\!\!/_c S^1$ is $\mathbb{C}^n$ with 0 replaced with $\mathbb{C}P^{n-1}$.

DEFINITION: In these assumptions, **symplectic blow-up** of radius $\lambda = \sqrt{c}$ of W in 0 is $(W \times \mathbb{C})/\!\!/_c S^1$. **Symplectic blow-up** of a symplectic manifold M is obtained by removing a symplectic ball W of radius $\sqrt{c}$ and gluing back a blown-up symplectic ball $(W \times \mathbb{C})/\!\!/_c S^1$.

Lagrangian blow-up

PROPOSITION: Consider the total space $(T^*\mathbb{C}P^1, \Omega_0)$ as a holomorhically symplectic manifold, and let $\pi^* \, T^*\mathbb{C}P^1 \to \mathbb{C}^2/\pm 1$ be the holomorphic contraction map. Let Ω_0 be the natural (constant coefficient) holomorphically symplectic form on $\mathbb{C}^2/\pm 1$, considered as a complex orbifold. **Then $\pi^*\Omega_0 = \Omega$.**

Proof: The ring of functions on $\mathbb{C}^2/\pm 1$ is generated by $a = x^2, b = xy, c = y^2$, and the holomorphic symplectic form is $dx \wedge dy = \frac{da \wedge db}{c}$. On $T^*\mathbb{C}P^1 = \operatorname{Tot}(\mathcal{O}(2))$, the functions π^*a, π^*b, π^*c have zero of second order, **hence da and db have zero of first order, and $\frac{da \wedge db}{c}$ is non-degenerate.** ■

DEFINITION: Let (B, ω) be a 4-dimensional symplectic ball, and $(B, \omega)/\pm$ its orbifold quotient. Using Darboux coordinates, we identify ω with $\operatorname{Re}\Omega$ (the real part of a holomorphic symplectic form). **The Lagrangian blow-up** $B/\pm 1$ is obtained by taking a holomorphic blow-up of $(B, \omega)/\pm$ with the standard complex structure. It is isomorphic to a neighbourhood of $\mathbb{C}P^1$ in $\operatorname{Tot}(T^*\mathbb{C}P^1)$. By the previous proposition, **$\pi^*\omega$ is non-degenerate, hence $B/\pm 1$ is equipped with a natural symplectic structure.**

REMARK: The blow-up sphere S^2 **is Lagrangian with respect to $\pi^*\omega$.**
REMARK: Similarly to the symplectic blow-up construction, this construction **can be applied to any 4-dimensional symplectic orbifold with double points,** resulting in a symplectic manifold with a bunch of Lagrangian spheres.

Lagrangian blow-down

THEOREM: (Weinstein's Lagrangian neighbourhood theorem) Let $X \subset M$ be a compact Lagrangian submanifold in (M, ω). Then **there exists a neighbourhood U of $X \subset M$ which is symplectomorphic to a neighbourhood of X in $X \subset T^*X$.**

Given a Lagrangian sphere S^2 in a symplectic 4-fold M, we chose its Weinstein neighbourhood W, symplectomorphic to an ε-neighbourhood of S^2 in T^*S^2, identify it with $(T^*\mathbb{C}P^1, \operatorname{Re}\Omega)$, where $\operatorname{Re}\Omega$ is the real part of the holomorphic symplectic form, and perform the (complex analytic) blow-down, resulting in holomorphic symplectic orbifold $(W_0, dx \wedge dy)$, obtained as a quotient of a neighbourhood of 0 in $\mathbb{C}^2$ by ± 1.
REMARK: Clearly, the manifolds W and W_0 **are naturally identified** outside of a small neighbourhood of S^2 in W and the orbifold point in W_0.

DEFINITION: The Lagrangian blow-down is obtained from M by removing $W \subset M$ and gluing W_0 in its place.

REMARK: By construction, the Lagrangian blow-up "is inverse" to the Lagrangian blow-down. Here "is inverse" means that we made many choices performing both operations; to show that these choices cancel each other, **we use the symplectic Teichmüller theory, which provides a way to speak of symplectic constructions functorially.**

Teichmüller space for symplectic structures

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. **Then $\Gamma(\Lambda^2 M)$ is a Fréchet vector space, and Symp a Fréchet manifold.**

DEFINITION: Consider the group of diffeomorphisms, denoted Diff, as a Fréchet Lie group, and denote its connected component ("group of isotopies") by Diff_0. The quotient group $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ is called **the mapping class group** of M.

DEFINITION: **Teichmüller space of symplectic structures on M** is defined as a quotient $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$.

Moser theorem

DEFINITION: Two symplectic structures are called **isotopic** if they lie in the same orbit of Diff_0.

DEFINITION: Define **the period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

REMARK: This means that Teich_s is smooth. However, **it is often non-Hausdorff.**

The proof is based on another theorem of Moser.

THEOREM: (Moser)
Let ω_t, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold S. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are diffeomorphic.**

Teichmüller space for Lagrangian submanifolds

DEFINITION: Fix a symplectic manifold (M, ω), and let $L \subset M$ be a Lagrangian submanifold with $b_1(L) := \dim H^1(L, \mathbb{R}) = 0$. Denote by Symp_L an infinite-dimensional space of symplectic forms ω on M such that $\omega|_L = 0$. We define **the Teichmüller space of pairs** (ω, L) as the quotient $\mathrm{Symp}_L / \mathrm{Diff}^0_L$, where Diff^0_L is the group of diffeomorphisms of M which preserve $L \subset M$, and Diff^0_L its connected component.

REMARK: The space $\mathrm{Symp}_L / \mathrm{Diff}^0_L$ is always smooth, but it is harder to prove this when $b_1(L) > 0$, because the period map is more difficult to define. Therefore, **we restrict ourselves to the case $b_1(L) = 0$.**

Moser theorem for the Teichmüller space of Lagrangian subvarieties

THEOREM: Let $V \subset H^2(M,\mathbb{R})$ be the space of all cohomology classes on M which vanish on L, and $\operatorname{Per}_L : \operatorname{Symp}_L / \operatorname{Diff}^0_L \longrightarrow V$ take (M,ω) to the cohomology class of ω. **Then** Per **is locally a diffeomorphism**.

Proof. Step 1: Let $\mathfrak{T}_L$ be the Teichmüller space of symplectic structures ω such that $\omega|_L$ is homologous to zero. By Moser's theorem, the period map $\mathfrak{T}_L \to V$ is locally a diffeomorphism. To prove the theorem, **it remains to show that the natural map** $\Psi : \operatorname{Symp}_L / \operatorname{Diff}^0_L \to \mathfrak{T}_L$ **is locally a locally a diffeomorphism.**

Step 2: We have assumed that L is Lagrangian in (M,ω). To prove that Ψ is locally surjective, it would suffice to prove that a small deformation $\omega_1 \in \mathfrak{T}_L$ of ω is isotopic to a form which is also Lagrangian on L. **This is equivalent to constructing a Lagrangian submanifold L_1 of (M,ω_1) which is smoothly isotopic to L.**

Moser for the Teichmüller space of Lagrangian subvarieties (2)

Step 3: Take a Weinstein neighbourhood U_L of L in (M,ω). Then $\omega_1 = \omega - d\theta$, where θ is a 1-form. Denote by L_1 the graph of θ in T^*L. For ω_1 sufficiently close to ω and θ sufficiently small, we may assume that L_1 belongs to U_L. On a graph of a map $\theta: L \longrightarrow T^*L$, the form ω is equal to $d\theta$, hence $L_1 \subset U_L$ is Lagrangian. **This shows that the $\Psi: \operatorname{Symp}_L / \operatorname{Diff}_L^0 \longrightarrow \mathfrak{T}_L$ is locally a surjection.**

Step 4: It remains to show that $\operatorname{Per}_L: \operatorname{Symp}_L / \operatorname{Diff}_L^0 \longrightarrow V$ is locally injective. This is proven using the same argument as Moser used: given a continuous family ω_t of cohomologous symplectic forms vanishing on L, we write the antiderivatives $\eta_t \in \Lambda^1(M)$ such that $\frac{d}{dt}\omega_t = d\eta_t$. Since $\omega_t|_L = 0$, the restrictions $\eta_t|_L$ are closed. They are exact because $b_1(L) = 0$, hence $\eta_r|_L = df_t$. We extend f_t to M smoothly and replace each η_t by $\eta_t - df_t$, obtaining $\eta_t|_X = 0$. Solving the equation $\operatorname{Lie}_{X_t}\omega_t = d(i_{X_t}\omega_t) = \frac{d}{dt}\omega_t$ as in Moser's proof, we obtain a flow of diffeomorphisms which takes ω_0 to ω_t; for this solution, we need $i_{X_t}\omega_t = \eta_t$. Since $\eta_t|_L = 0$, and L is Lagrangian with respect to ω_t, the vector field X_t remains tangent to L. ■

COROLLARY: Assume that L, L' are Lagrangian submanifolds in (M,ω), and $\varphi \in \operatorname{Diff}_0(M)$ a smooth isotopy such that $\varphi(L) = L'$. **Then (ω, L) and $(\varphi^*\omega, L)$ represent the same point in Teich_L if and only if L and L' are Lagrangian isotopic in (M,ω).**

Teichmïler space and Lagrangian blow-ups and blow-downs

REMARK: From now on, we will focus on just one example, namely the K3 surface and the orbifold K3, obtained by a contraction of a smooth rational curve. **This is done do simplify the conventions**, everything works in a general situation.

THEOREM: Let L be a Lagrangian 2-sphere in a K3 surface M, and Teich_L the corresponding Teichmüller space of symplectic structures vanishing on L. Consider an orbifold K3 surface $\check{M}$ obtained from the Lagrangian blow-down, and let $\operatorname{Teich}_{\check{M}}$ be its symplectic Teichmüller space. **Then the Lagrangian blow-down defines a diffeomorphism $\operatorname{Teich}_L \longrightarrow \operatorname{Teich}_{\check{M}}$.**

Proof: Clearly, the Lagrangian blow-down is (up to choices made) inverse to the Lagrangian blow-up. Therefore, **it suffices to prove that both constructions are independent from all the choices we made** (see the next slide).

Teichmüller space and Lagrangian blow-ups

THEOREM: Let $\operatorname{Teich}_{\check{M}}$ be a 4-dimensional symplectic orbifold with a single double point, and (M, L, ω) its Lagrangian blow-up. **Then the corresponding point in Teich_L is independent from the choices we made.**

Proof: Lagrangian blow-up is determined by a germ of the Darboux coordinates in a neighbourhood of the orbifold point o. For any two choices A, B of Darboux coordinates around o, there is a germ of a flow φ_t of symplectomorphisms in a neighbourhood of o taking a germ of A to a germ of B. Since $b_1(A) = 0$, the diffeomorphisms φ_t are Hamiltonian. Extending the corresponding Hamiltonian vector fields to M, we obtain a flow Φ_t Hamiltonian symplectomorphisms of M which carries A to B. This defines a smooth family of symplectic structures on the Lagrangian blow-up, which are all homologous, and connect the first blow-up to the second; by Moser's theorem, **this implies that these two blow-up manifolds correspond to the same point in Teich_L.** ■

Teichmїler space and Lagrangian blow-downs

THEOREM: Let (M, L, ω) be a 4-manifold with a Lagrangian sphere L, and $(\check{M}, \check{\omega})$ its Lagrangian blow-down. **Then the corresponding point in $\operatorname{Teich}_{\check{M}}$ is independent from the choices we made.**

Proof: The Lagrangian blow-down of a Lagrangian sphere S in a symplectic 4-manifold is determined by a germ of a Weinstein neighbourhood of S; same as in the previous proof, we need to show that any two such germs are related by a smooth family of Hamiltonian symplectomorphisms. By definition, any two Weinstein neighbourhoods of S have germs in S which are symplectomorphic; these germs are Hamiltonian symplectomorphic, because $b_1(S) = 0$, hence any symplectomorphic vector field on a Weinstein neighbourhood is Hamiltonian. This implies that **any two choices of Lagrangian blow-down are related by a smooth family of cohomologous symplectic structures on $\check{M}$, hence by Moser's theorem they give the same point in $\operatorname{Teich}_{\check{M}}$.** ■

Donaldson's conjecture is false for orbifold K3 surfaces

CONJECTURE: (Donaldson) All symplectic structures on a K3 surface are compatible with a Kähler structure.

This conjecture is still open. However, an orbifold version of this conjecture is false.

THEOREM: There exists an orbifold symplectic K3 surface M with a single double point **not admitting an orbifold Kähler structure**.

Proof. Step 1: As follows from a result of Amerik-V. (2015), the Teichmüller space of symplectic structures compatible with a hyperkähler structure is Hausdorff and connected (the same argument works for Kähler K3 orbifolds). Suppose, on contrary, that any orbifold symplectic K3 surface with a single double point admits a Kähler structure. This would imply that the Teichmüller space Teich_L is connected, **and, indeed, that in each homotopy class of 2-spheres on a K3 surface there exists at most one (up to a Lagrangian isotopy) Lagrangian sphere.**

Step 2: This is impossible, by a result of P. Seidel (2000): **for some Lagrangian 2-sphere in a symplectic K3 surface there exists infinitely many Lagrangian spheres in the same smooth isotopy class,** which are not Lagrangian isotopic. ■

Special Lagrangian submanifolds

DEFINITION: Let (M, I, ω) be a symplectic manifold equipped with a compatible almost complex structure, $\dim M = 2n$, and Φ a non-degenenerate, closed $(n,0)$-form which satisfies $|\Phi| = const$. A **phase** of a Lagrangian submanifold $S \subset M$ is the complex-valued function $\frac{\Phi|_S}{\mathrm{Vol}_S}$ on S. A Lagrangian submanifold $S \subset M$ is called **special Lagrangian** if it has constant phase.

THEOREM: Let $S \subset M$ be a Lagrangian sphere in a K3 surface, and $\pi : (M, \omega) \longrightarrow (\check{M}, \breve{\omega})$ the corresponding Lagrangian blow-down map. **Then S is Lagrangian isotopic to a special Lagrangian sphere if and only if the symplectic form $\breve{\omega}$ is of Kähler type.**

Proof: Later today.

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold. A complex manifold $S \subset M$ is **holomorphic Lagrangian** if $\Omega|_S = 0$ and $\dim M = 2 \dim S$.

REMARK: On a hyperkähler manifold, **"special Lagrangian" follows from "holomorphic Lagrangian"**; in real dimension 4, the converse is true as well, if one varies the complex structure in the quaternions.

Holomorphic Lagrangian submanifolds

The condition of being a complex submanifold **is in fact implied by being Lagrangian with respect to the symplectic forms** $\operatorname{Re}\Omega$ **and** $\operatorname{Im}\Omega$**.**

PROPOSITION: ("Hitchin's lemma")
Let (M, I, Ω) be a holomorphically symplectic manifold, and $S \subset M$ a real submanifold, $\dim M = 2\dim S$. **Then S is holomorphic Lagrangian if and only if $\Omega|_S = 0$.**

Corollary 1: Let (M, I, J, K, g) be a hyperkähler manifold of real dimension $4n$, considered as a Kähler manifold (M, I, ω_I), and $\Phi := \Omega^n \in \Lambda^{2n,0}(M, I)$ the corresponding holomorphic volume form. Consider a Lagrangian submanifold $S \subset (M, \omega_I)$, such that $a\omega_J + b\omega_K|_S = 0$, for some real numbers $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$. **Then S is special Lagrangian, and, moreover, it is holomorphic with respect to the complex structure $-aK + bJ$.**

Proof: The form $\Omega_1 := \omega_I + \sqrt{-1}\,(a\Omega_J + b\Omega_K)$ is holomorphic symplectic on $(M, -aK + bJ)$, hence $\Omega_1|_S = 0$ By Hitchin's lemma, S is a complex submanifold of $(M, -aK + bJ)$, and therefore it is holomorphic Lagrangian. It is special Lagrangian, because in the 3-dimensional space $\langle \omega_I, \omega_J, \omega_K \rangle$ there is a basis, where two forms restrict to S as zero, and the third is the Kähler form of $(M, -aK + bJ)$. Its top power restricted to a complex submanifold is always proportional to its Riemannian volume, hence the phase of S is constant. ■

Special Lagrangian submanifolds in real dimension 4

Corollary 2: Let M be a 4-dimensional hyperkähler manifold, $\omega_I, \omega_J, \omega_K$ a triple of symplectic structures and $S \subset M$ a Lagrangian submanifold of (M, ω_I). **Then S is special Lagrangian if and only if for some real numbers $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, the submanifold S is complex analytic with respect to $-aK + bJ$.**

Proof: If S is complex analytic with respect to $-aK + bJ$, the corresponding holomorphic symplectic form vanishes on S, because $\dim_{\mathbb{C}} S = 1$, and **by Corollary 1 it is special Lagrangian.** Conversely, for any special Lagrangian submanifold $S \subset (M, I)$, the constant phase condition $\Omega|_S = const\, \mathrm{Vol}_g$ implies that the restriction map $\langle \omega_J, \omega_K \rangle$ has 1-dimensional kernel, hence S is Lagrangian with respect to ω_I and $a\omega_J + b\sqrt{-1}\, \omega_K$, for some $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. **By Hitchin's lemma, S is holomorphic with respect to $-aK + bJ$.** ■

Symplectic K3 orbifolds of Kähler type

Further on, we will use the following elementary observation.

CLAIM: Let $(\check{M}, I)$ be a complex orbifold of K3 type, and $\check{\Omega}$ its holomorphic symplectic form. **Then $\operatorname{Re}\check{\Omega}$ is a symplectic structure of Kähler type.**

Proof: Using an orbifold version of Calabi-Yau theorem, **we choose a hyperkähler structure (I, J, K, g) on $\check{M}$, such that $\check{\Omega} = \omega_J + \sqrt{-1}\,\omega_K$.** Then $\operatorname{Re}\check{\Omega} = \omega_J$ is a Kähler form of $(\check{M}, J)$. ■

DEFINITION: Let M be a K3 orbifold, and ω an orbifold symplectic structure. We say that ω **is of Kähler type** if there exists an orbifold Kähler structure such that ω is its Kähler form.

Special Lagrangian spheres and symplectic structures on orbifolds

THEOREM: Let $S \subset M$ be a Lagrangian sphere in a K3 surface, and $\pi : (M,\omega) \longrightarrow (\check M,\check\omega)$ the corresponding Lagrangian blow-down map. **Then S is Lagrangian isotopic to a special Lagrangian sphere if and only if the symplectic form $\check\omega$ is of Kähler type.**

Proof. Step 1: Let $S \subset M$ be a special Lagrangian sphere. Since $S \subset (M,\omega)$ is Lagrangian, its self-intersection is -2. Without restricting the generality, we may assume that $S \subset (M,K)$ is a complex curve, that is, $a=0, b=1$ in the notation of Corollary 2. Consider the complex analytic blow-down map $\pi: (M,K) \mapsto (\check M, K)$, taking M to a complex orbifold with a double point. Let $\Omega_K := \omega_I + \sqrt{-1}\,\omega_J$ be a holomorphic symplectic form on (M,K). Then $\Omega_K = \pi^*\check\Omega_K$, hence $\omega_I = \operatorname{Re}\Omega_K = \pi^*\operatorname{Re}\check\Omega_K$. **This identifies $(\check M, \operatorname{Re}\Omega_K)$ and the Lagrangian blow-down of (M,ω_I).** However, $(\check M, \operatorname{Re}\Omega_K)$ is a complex orbifold, and the symplectic form $\operatorname{Re}\check\Omega_K$ is of Kähler type by the previous claim. **This proves that the Lagrangian blow-down of (M,ω_I) is of Kähler type.**

Step 2: Conversely, assume that $(\check M,\check\omega)$ is an orbifold symplectic K3 of Kähler type with one double point, and $(M,\pi^*\check\omega)$ its Lagrangian blow-up. Since the blow-up map is holomorphic on $(\check M,K)$, the exceptional sphere of the projection $\pi: M \longrightarrow \check M$ is holomorphic in (M,K). **By Corollary 1, it is special Lagrangian in (M,I).** ■

Lagrangian spheres which are not Hamiltonian isotopic to special Lagrangian

THEOREM: Let (M, ω) be a symplectic K3 surface of Kähler type. **Then, for an appropriate choice of ω, there exists a countably many Lagrangian 2-spheres in the same homotopy class which are not isotopic to a special Lagrangian sphere.**

Proof: Using Seidel's thorem, we obtain that **there exists a countably many pairwise non-Lagrangian-isotopic Lagrangian spheres S_i in the same homotopy class such that the corresponding Lagrangian blow-down give a K3-type orbifolds which are not of Kähler type.** By the previous theorem, all these S_i are not isotopic to a special Lagrangian sphere.

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