Hilbert metrics and averaging

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Mapping class group action on cohomology

DEFINITION: Let M be a manifold, Diff its diffeomorphism group, Diff_0 its connected component. The group $\Gamma \subset \operatorname{Diff} / \operatorname{Diff}_0$ is called **the mapping class group** of M.

REMARK: When $\pi_1(M)$ is nilpotent and $\dim M > 4$, **Sullivan has shown that Γ is an arithmetic group (that is, commensurable with a group of integer points in a Lie group).**

EXAMPLE: For a torus $T = (S^1)^n$, Γ acts on $H^*(T)$ as $GL(n, \mathbb{Z})$, for a hyperkähler manifold it is a finite index subgroup in $O(H^2(M, \mathbb{Z}), q)$, where q is BBF form.

Γ-invariant pairing

QUESTION: Let M be a compact n-manifold. **Does there exist a Γ-invariant non-degenerate pairing on $H^i(M)$, for all i?**

REMARK: When $n = 2i$, it is Poincare pairing.

EXAMPLE: **When M is a torus, and $i \neq n/2$, such a pairing does not exist,** because there are no $SL(n, \mathbb{Z})$-invariant tensors in $\Lambda^i(\mathbb{R}^n) \otimes \Lambda^i(\mathbb{R}^n)$.

EXAMPLE: When M is hyperkähler, such a pairing exists, because representations of $O(n)$ admit a non-degenerate bilinear form.

QUESTION: **Can we define this pairing in a more functorial way?**

Averaging

DEFINITION: G is called **a reductive Lie group** if all its finite-dimensional represenations are semisimple.

REMARK: A group is reductive **if and only if its Lie algebra is a direct sum of semisimple Lie algebra and abelian.** It follows immediately from the Levi decomposition theorem.

REMARK: A compact Lie group is clearly reductive; conversely, **a complexification of a reductive Lie group always has a compact real form.**

EXERCISE: Prove these statements.

DEFINITION: Let G be a reductive Lie group, and V its representation, and V_{inv} the space of G-invariant vectors. The G-invariant projection $\operatorname{Av} : V \longrightarrow V_{\text{inv}}$ is called **the averaging map**.

REMARK: When G is compact, Av is really averaging with respect to Haar measure.

Averaging and convex hulls

QUESTION: Let V be a representation of a reductive Lie group G, Gv an orbit of $v \in V$, and Hull(Gv) its convex hull. **Is it true that** Av$(v) \in$ Hull(Gv)**?**

EXAMPLE: Let $V = \mathbb{R}^{p+q}$ be the fundamental representation of $G = SO(p,q)$, and h a positive definite bilinear symmetric form on V. Then all points of Hull(Gv) are positive definite forms on V, however, Schur's lemma implies that none of them is G-invariant. **Therefore,** Av$(v) \notin$ Hull(Gv)**.**

EXAMPLE: Let $V = \mathbb{R}^{n+1}$ be the fundamental representation of $G = SO(1,n)$, and Pos$(V) \subset V$ its positive cone, that is, one of the components of $\{v \in V \ \ | \ \ (v,v) > 0\}$. By Cauchy-Schwarz inequality, Pos(V) is convex, however, Av$(v) = 0$ for any $v \in V$. **Therefore,** Av$(v) \notin$ Hull(Gv)**.**

REMARK: Let $K \subset \mathbb{R}^n$ be a convex set, and $W \subset \mathbb{R}^n$ the smallest affine subspace containing K. The set of **interior points** of K is the union of all open subsets of W contained in K.

The main result of today's lecture:

THEOREM: Let V be a representation of a reductive Lie group G, Gv an orbit of $v \in V$, and Hull(Gv) its convex hull. Assume that Av$(v) \notin$ Hull(Gv). **Then for any interior point** $w \in$ Hull(Gv)**, its stabilizer** St$_G(w)$ **is compact.**

Averaging and convex hulls: motivation

It is motivated by the following question (still unsolved).

QUESTION: Let M be a hyperkähler manifold, Γ its mapping class group, and $G \subset \operatorname{Aut}(H^*(M))$ its Zariski closure. Let ω be a Kähler class, and consider the following pairing on $H^{2d}(M)$:

$$q_\omega(x, y) = \int_M x \wedge y \wedge \omega^{2n-d},$$

where $2n = \dim_{\mathbb{C}} M$. **Is $\operatorname{Av}_G(q_\omega)$ non-degenerate?** For $H^2(M)$ it is non-degenerate, and **this is how the BBF form is obtained.**

Hyperbolic metric

LEMMA: Let $U \subset \mathbb{R}P^1$ be an interval, and $\operatorname{St}_{PGL(2)} U$ its stabilizer. **Then $\operatorname{St}_{PGL(2)} U = PSO^+(1,1)$,** where $SO^+(1,1)$ denotes the connected component.

Proof: Let $h \in \operatorname{Sym}^2(\mathbb{R}^2)$ be a bilinear symmetric form of signature (1,1) vanishing in the ends of the interval U. The form h is defined by this condition up to a scalar multiplier. **Therefore, $\mathbb{R}h$ is fixed by the group $\operatorname{St}_{PGL(2)}$, which gives $\operatorname{St}_{PGL(2)} U = PSO^+(1,1)$.** ■

DEFINITION: Let V be a vector space equipped with a form of signature $(1,n)$, and $\mathbb{P}\operatorname{Pos}(V)$ projectivisation of its positive cone. Clearly, a $SO^+(1,n)$-invariant metric h on $\mathbb{P}\operatorname{Pos}(V)$ is unique, up to a constant multiplier. We call $(\mathbb{P}\operatorname{Pos}(V), h)$ **hyperbolic space**, or **Lobachevsky space**, and denote it by $\mathbb{H}$.

REMARK: The constant can be fixed by any reasonable convention, e. g. by identifying $\mathbb{H}$ with the space of unit vectors in Pos and considering the induced metric.

Hilbert metric

DEFINITION: A subset $U \subset \mathbb{R}P^n$ is called **convex** if it is an image of a convex subset in $\mathbb{R}^{n+1} \backslash 0$.

DEFINITION: Hilbert metric on a convex set $U \subset \mathbb{R}P^n$ is a Finsler metric which restricts to a hyperbolic metric on each straight interval $\mathbb{R}P^1 \cap U$.

REMARK: Since $\operatorname{St}_{PGL(2)} I = PSO^+(1,1)$ for an interval $I \subset \mathbb{R}P^1$, Hilbert metric is invariant with respect to projective automorphisms.

DEFINITION: Let U, W be open subsets in projective spaces, and $\varphi : U \longrightarrow W$ a map which maps straight intervals to straight intervals and acts linearly on the corresponding open cones in $\mathbb{R}^2$. Then φ is called **projective map**.

REMARK: Clearly, **projective maps are 1-Lipschitz with respect to the Hilbert metric (prove it).**

Projective automorphisms of hyperbolic subsets

DEFINITION: Let $U\subset \mathbb{R}P^n$ a convex subset, obtained as a projectivization of a convex cone $\tilde U$. Then $\tilde U$ is called **the convex cone associated with U**. Clearly, it is projected to U with fiber $\mathbb{R}^{>0}$.

LEMMA 1: Let $U\subset \mathbb{R}P^n$ be a convex subset, $\tilde U$ its convex cone, and $v\in\tilde U$ any vector. Consider the group G of projective automorphisms of U fixing v. **Then G is compact.**

Proof: Clearly, G is the group of projective automorphisms of U fixing the line l_v passing through v. Then it acts on U by isometries. However, the group of isometries of a Finsler manifold fixing a point l_v is compact **(prove it as an exercise).** ■

Proof of main theorem

THEOREM: Let V be a representation of a reductive Lie group G, Gv an orbit of $v \in V$, and Hull(Gv) its convex hull. Assume that Av$(v) \notin$ Hull(Gv). **Then for any interior point $w \in$ Hull(Gv), its stabilizer St$_G(w)$ is compact.**

Proof: Substracting Av(v), we can always assume that Av$(v) = 0$. Let $\tilde{U}$ be the set of interior points of Hull(Gv). Since $\tilde{U}$ is a convex, open cone not containing 0, it can be separated from any $z \notin \tilde{U}$ by a hyperplane (Hahn-Banach theorem). Therefore, U is an intersection of half-spaces, and **its projectivization U is convex.** Then Lemma 1 implies that St$_G(w)$ is compact. ■