Deformations of complex structures given by differential forms

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Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

REMARK: The "usual definition": complex structure is an atlas on a manifold with differentials of all transition functions in $GL(n,\mathbb{C})$.

THEOREM: (Newlander-Nirenberg)
These two definitions are equivalent.

REMARK: An almost complex structure I **is uniquely determined by a subbundle $B \subset TM \otimes_{\mathbb{R}} \mathbb{C}$** such that $TM \otimes_{\mathbb{R}} \mathbb{C} = B \oplus \overline{B}$. Then we write $I = \sqrt{-1}$ on B and $I = -\sqrt{-1}$ on $\overline{B}$.

Null-space of a form and and Cartan's formula

DEFINITION: Let Ω be a differential form on M. The **kernel**, or **the null-space** $\ker(\Omega) \subset TM$ of Ω is the space of all vector fields $X \in TM$ such that the contraction $i_X(\Omega)$ vanishes.

Theorem 1: Let Ω be a differential p-form on M, $d\Omega = 0$. **Then for any $X, Y \in \ker(\Omega)$, one has $[X, Y] \in \ker(\Omega)$.**

Proof. Step 1: Let $X, X_1 \in \ker(\Omega)$, and $X_2, ..., X_p$ any vector fields. Cartan's formula implies that $\operatorname{Lie}_X(\Omega) = d(i_X(\Omega)) + i_X(d\Omega) = 0$, hence $\operatorname{Lie}_X(\Omega) = 0$.

Step 2: $\operatorname{Lie}_X(\Omega)(X_1, ..., X_p) = \operatorname{Lie}_X(\Omega(X_1, ..., X_p)) - \sum_{i=1}^p \Omega(X_1, ..., [X, X_i], ...X_p)$. All terms of this sum, except $\Omega([X, X_1], X_2, ..., X_p)$, vanish, because $X_1 \in \ker(\Omega)$. Since $\operatorname{Lie}_X(\Omega) = 0$, we have $\Omega([X, X_1], X_2, ..., X_p) = 0$ for all $X_2, ..., X_p$. Therefore, $[X, X_1] \in \ker(\Omega)$. ■

Corollary 1: Suppose that $\Omega \in \Lambda^2(M, \mathbb{C})$ be a closed p-form such that $\ker(\Omega) \cap T_{\mathbb{R}}M = 0$ and $\ker(\Omega) \oplus \overline{\ker(\Omega)} = T_{\mathbb{C}}M$. Define $I : TM \longrightarrow TM$ by $I|_{\ker(\Omega)} = -\sqrt{-1}$ and $I|_{\overline{\ker(\Omega)}} = \sqrt{-1}$. **Then I defines a complex structure.**

Proof: I is by construction real and satisfies $I^2 = -1$. The corresponding eigenspace $T^{1,0}M$ coincides with $\ker(\Omega)$, and Theorem 1 implies that I is integrable. ■

Holomorphically symplectic manifolds

DEFINITION: Let (M,I) be a complex manifold, and $\Omega \in \Lambda^2(M,\mathbb{C})$ a differential form. We say that Ω is **non-degenerate** if $\ker \Omega \cap T_{\mathbb{R}}M = 0$. We say that it is **holomorphically symplectic** if it is non-degenerate, $d\Omega = 0$, and $\Omega(IX,Y) = \sqrt{-1}\,\Omega(X,Y)$.

REMARK: The equation $\Omega(IX,Y) = \sqrt{-1}\Omega(X,Y)$ means that Ω **is complex linear with respect to the complex structure on** $T_{\mathbb{R}}M$ **induced by** I.

REMARK: Consider the Hodge decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ (decomposition according to eigenvalues of I). Since $\Omega(IX,Y) = \sqrt{-1}\,\Omega(X,Y)$ and $I(Z) = -\sqrt{-1}\,Z$ for any $Z \in T^{0,1}(M)$, we have $\ker(\Omega) \supset T^{0,1}(M)$. Since $\ker \Omega \cap T_{\mathbb{R}}M = 0$, real dimension of its kernel is at most $\dim_{\mathbb{R}} M$, giving $\dim_{\mathbb{R}} \ker \Omega = \dim M$. **Therefore,** $\ker(\Omega) = T^{0,1}M$.

COROLLARY: Let Ω be a holomorphically symplectic form on a complex manifold (M,I). **Then I is determined by Ω uniquely.**

Let's define complex structures in terms of complex-valued 2-forms!

Holomorphically symplectic forms and complex structures

CLAIM: Let M be a smooth $2n$-dimensional manifold. **Then there is a bijective correspondence between the set of almost complex structures, and the set of sub-bundles $T^{0,1}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $\dim_{\mathbb{C}} T^{0,1}M = n$ and $T^{0,1}M \cap TM = 0$** (the last condition means that there are no real vectors in $T^{1,0}M$, that is, that $T^{0,1}M \cap T^{1,0}M = 0$).

Proof: Set $I\big|_{T^{1,0}M} = \sqrt{-1}$ and $I\big|_{T^{0,1}M} = -\sqrt{-1}$. ■

Theorem 2: Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a smooth, complex-valued, non-degenerate 2-form on a $4n$-dimensional real manifold. Assume that $\Omega^{n+1} = 0$. Consider the bundle

$$T^{0,1}_{\Omega}(M) := \{v \in TM \otimes \mathbb{C} \ \ | \ \ \Omega \lrcorner v = 0\}.$$

Then $T^{0,1}_{\Omega}(M)$ satisfies assumptions of the claim above, hence **defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable.**

Proof: Rank of Ω is $2n$ because $\Omega^{n+1} = 0$ and it is non-degenerate. Then $\ker \Omega \oplus \overline{\ker \Omega} = T_{\mathbb{C}} M$. Now Theorem 2 follows from Theorem 1. ■

Hyperkähler manifolds

DEFINITION: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators $I,J,K : TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$. Suppose that I, J, K are Kähler. Then (M,I,J,K,g) is called **hyperkähler**.

CLAIM: A hyperkähler manifold (M,I,J,K) is **holomorphically symplectic** (equipped with a holomorphic, non-degenerate 2-form). Recall that M is equipped with 3 symplectic forms ω_I, ω_J, ω_K.

LEMMA: The form $\Omega := \omega_J + \sqrt{-1}\omega_K$ **is a holomorphic symplectic 2-form on** (M,I)**.** ■

THEOREM: (Calabi-Yau, 1978) Let M be a compact, holomorphically symplectic Kähler manifold. Then M **admits a hyperkähler metric,** which is uniquely determined by the cohomology class of its Kähler form ω_I.

For compact manifolds, **"hyperkähler" is essentially synonymous with "holomorphically symplectic of Kähler type"**.

Holomorphic Lagrangian fibrations

DEFINITION: Let (M, Ω) be a holomorphically symplectic manifold, $\dim_{\mathbb{C}} M = 2n$. A complex subvariety $Z \subset M$ is called **holomorphically Lagrangian** if $\Omega|_Z = 0$ in all smooth points of Z and $\dim Z = n$.

DEFINITION: **A holomorphic Lagrangian fibration** is a holomorphic map $f: M \longrightarrow X$ with all fibers holomorphic Lagrangian.

REMARK: **Nota bene:** **Neither X nor fibers of f need to be non-singular:** the definition makes sense in singular situation.

DEFINITION: **(1,1)-form** on a complex manifold is a differential form which satisfies $\omega(IX, Y) = -\omega(X, IY)$ (same Hodge type as the Hermitian forms).

Matsushita Theorem

THEOREM: (Matsushita, 1997)
Let $\pi: M \longrightarrow X$ be a surjective holomorphic map from a hyperkähler manifold M to X, whith $0 < \dim X < \dim M$. **Then $\dim X = 1/2 \dim M$, and the fibers of π are holomorphic Lagrangian** (this means that the symplectic form vanishes on $\pi^{-1}(x)$).

DEFINITION: Such a map is called **holomorphic Lagrangian fibration**.

REMARK: The base of π is conjectured to be $\mathbb{C}P^n$ if it is normal. Hwang (2007) proved that $X \cong \mathbb{C}P^n$, if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as $\mathbb{C}P^n$.

REMARK: The base of π has a natural flat connection on the smooth locus of π. **The combinatorics of this connection can be used to determine the topology of M** (Strominger-Yau-Zaslow, Kontsevich-Soibelman).

If we want to learn something about M, it's recommended to start from a holomorphic Lagrangian fibration (if it exists).

Degenerate twistor deformations

THEOREM: Let (M,Ω) be a holomorphically symplectic manifold, $\dim_{\mathbb{C}} M = 2n$, and $f: M \longrightarrow X$ a holomorphic Lagrangian fibration. Consider a $(2,0)+(1,1)$-form $\eta \in \Lambda^2(X,\mathbb{C})$. Then (a) $\Omega_\eta := \Omega + f^*\eta$ is a **non-degenerate form which satisfies $\Omega_\eta^{n+1} = 0$, hence defines an almost complex structure.** (b) If, moreover, $d\eta = 0$, this almost complex structure is integrable.

Proof: (b) follows from Theorem 2 and (a) immediately. Relation $\Omega_\eta^{n+1} = 0$ is proving by writing Ω and η in a basis dp_i, dq_i such that $\Omega = \sum_i dp_i \wedge dq_i$, coordinates on X are p_i, and

$$\eta = \sum \beta_{ij} dp_i \wedge dp_j + \sum \alpha_i dp_i \wedge d\overline{p}_i,$$

where $\alpha_i, \beta_{ij} \in C^\infty X$. Non-degeneracy follows immediately, because (2,0)-part of Ω_η is $\Omega + \eta^{2,0}$, and it is non-degenerate. ■

DEFINITION: Let (M,Ω) be a holomorphically symplectic manifold, $\dim_{\mathbb{C}} M = 2n$, $f: M \longrightarrow X$ a holomorphic Lagrangian fibration, and $\eta \in \Lambda^{1,1}(X,\mathbb{C})$ a closed (1,1)-form. Let $\Omega_{t\eta} := \Omega + t\eta$, where $t \in \mathbb{C}$, and let I_t be the complex structure associated with t. The family of complex structures I_t is called **a degenerate twistor deformation** of M.

Degenerate twistor deformations: their properties

1. If η is exact, this deformation is trivial by Moser's theorem.

2. When M is compact and of Kähler type, M is hyperkähler. In this case I_t is a limit of *twistor deformations.* **The manifolds (M, I_t) are projective for $t \in S$, where S is a dense, countable family $S \subset \mathbb{C}$, and non-algebraic for $t \notin S$. It is unknown if (M, I_t) is Kähler for all $t \in \mathbb{C}$.**

3. Each manifold (M, I_t) is equipped with a holomorphic Lagrangian fibration $f_t : (M, I_t) \longrightarrow X$. **The fibers and the base of f_t are isomorphic for all t.**

4. This deformation has a number-theoretic interpretation ("Tate-Shafarevich deformation").

Sections of holomorphic Lagrangian fibrations

THEOREM: (Hitchin) Let (M,Ω) be a holomorphically symplectic manifold, and $Z\subset M$ a subvariety which is Lagrangian with respect to $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$. **Then Z is a complex subvariety.**

Proof: Let α be the determinant vector of Z, and $L_J, L_K, \Lambda_J, \Lambda_K$ elements of Hodge triples associated with J, K. Then $[L_J,\Lambda_k]\alpha=0$. However, $[L_J,\Lambda_k]$ acts on $\Lambda^{p,q}(M)$ as $(p-q)\sqrt{-1}$, which implies that α is of Hodge type (n,n).
■

COROLLARY: Let (M,Ω) be a holomorphic symplectic manifold, $\pi: M\longrightarrow B$ a holomorphic Lagrangian fibration, and $B_1\subset M$ a section of π. Suppose that $\Omega\big|_{TB_1}$ is of Hodge type $(1,1)+(2,0)$ with respect to the natural complex structure on $B_1=B$. **Then there exists a degenerate twistor deformation given by $\Omega+\pi^*\eta$, such that B_1 is holomorphic**. ■

Sections of holomorphic Lagrangian fibrations (2)

THEOREM: Let (M,Ω) be a holomorphic symplectic manifold, $\pi: M \longrightarrow B$ be a holomorphic Lagrangian fibration, and $B_1 \subset M$ a smooth section of π. Consider the form $\Omega\Big|_{B_1}$ as a form on B. **Then it has type $(2,0)+(1,1)$.**

REMARK: Then **it follows that B_1 becomes holomorphic after a degenerate twistor deformation.**

Proof: Consider the map $\Omega: TM \longrightarrow T^*M$. Since π is a Lagrangian fibration, $\Omega(T_\pi M) \subset \pi^*(\Lambda^1 B)$, where $T_\pi M$ denotes vector tangent to the fibers of π **("vertical tangent vectors")** Since the (1,0)-component of $\pi^{-1}(T^{0,1}B)$ is vertical, one has

$$\Omega(\pi^{-1}(T^{0,1}B)) \subset \pi^*\Lambda^{1,0}B.$$

This implies that $\Omega\Big|_{B_1}$ pairs (0,1) vectors to (1,0)-vectors, and has type $(1,1)+(2,0)$. ■