

Any component of moduli of polarized hyperkahler manifolds is dense in its deformation space

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Moduli spaces and automorphic forms

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Plan of the talk

This is a joint work with Sasha Anan'in (UNICAMP)

1. Introduce hyperkähler manifolds and their moduli. Define **the birational moduli space as a quotient of a Teichmüller space** $\mathbb{P}_{\text{Per}} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ **by an arithmetic group** Γ_I .
2. Define the polarized Teichmüller space as a divisor in \mathbb{P}_{Per} .
3. Show that **its image in the moduli space** $\mathbb{P}_{\text{Per}}/\Gamma_I$ **is dense.**
4. Explain the background story: why is this interesting.

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2, 0)$ -form.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Teichmüller space and the mapping class group

Definition: Let M be a compact complex manifold, and $\text{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\text{Teich}}$ the space of complex structures on M , and let $\text{Teich} := \widetilde{\text{Teich}} / \text{Diff}_0(M)$. We call it **the Teichmüller space**.

Remark: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

Definition: Let $\text{Diff}_+(M)$ be the group of oriented diffeomorphisms of M . We call $\Gamma := \text{Diff}_+(M) / \text{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on M** is a connected component of Teich / Γ .

Remark: This terminology is **standard for curves**.

REMARK: For hyperkähler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta, \eta)^n$, for some primitive integer quadratic form q on $H^2(M, \mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. It is defined by the Fujiki's relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta, \eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \frac{n-1}{n} \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n \right) \left(\int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1} \right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: q has signature $(b_2 - 3, 3)$. It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega \rangle$, where ω is a Kähler form.

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geq 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M, \mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\text{Diff}_+(M)/\text{Diff}_0 \rightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0 .**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then

- (i) $\Gamma_0|_{H^2(M, \mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M, \mathbb{Z}), q)$.**
- (ii) The map $\Gamma_0 \rightarrow O(H^2(M, \mathbb{Z}), q)$ **has finite kernel.**

The period map

Remark: For any $J \in \text{Teich}$, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $P : \text{Teich} \rightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $P : \text{Teich} \rightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: P maps Teich into an open subset of a quadric, defined by

$$\text{Per} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

It is called **the period space** of M .

REMARK: $\text{Per} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

DEFINITION: The space $\text{Teich}_b := \text{Teich} / \sim$ is called **the birational Teichmüller space** of M .

THEOREM: **The period map** $\text{Teich}_b \xrightarrow{\text{Per}} \mathbb{P}er$ **is an isomorphism**, for each connected component of Teich_b .

DEFINITION: Let M be a hyperkaehler manifold, Teich_b its birational Teichmüller space, and Γ the mapping class group. The quotient Teich_b / Γ is called **the birational moduli space** of M .

THEOREM: Let (M, I) be a hyperkähler manifold, and W a connected component of its birational moduli space. **Then W is isomorphic to $\mathbb{P}er / \Gamma_I$, where $\mathbb{P}er = SO(b_2 - 3, 3) / SO(2) \times SO(b_2 - 3, 1)$ and Γ_I is an arithmetic group in $O(H^2(M, \mathbb{R}), q)$.**

A CAUTION: Usually “the global Torelli theorem” is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M, \mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, **it is false**.

Polarized Teichmüller space

THEOREM: (Demailly, Paun, Huybrechts, Boucksom) Let M be a hyperkähler manifold, such that all integer $(1,1)$ -classes satisfy $q(\nu, \nu) \geq 0$. Then its Kähler cone is one of two connected components of the set $K := \{\nu \in H^{1,1}(M, \mathbb{R}) \mid q(\nu, \nu) > 0\}$.

REMARK: $H^{1,1}(M, \mathbb{R})$ is an orthogonal complement to $\langle \Omega, \overline{\Omega} \rangle$, where Ω is the cohomology class of holomorphic symplectic form.

DEFINITION: Define $\text{Teich}_\eta \subset \text{Teich}$ to be the set of all I with $l := \text{Per}(I) \in \text{Per} \subset \mathbb{P}H^2(M, \mathbb{C})$ satisfying $l \in \langle \Omega, \overline{\Omega} \rangle^\perp$.

COROLLARY: Let $\eta \in H^2(M, \mathbb{Z})$ be a vector which satisfies $q(\eta, \eta) > 0$. Then, **for a general $I \in \text{Teich}_\eta$, the class η is ample on (M, I) .**

DEFINITION: We call Teich_η **the polarized Teichmüller space** of M . It is a divisor in Teich .

REMARK: η is not necessarily ample in **non-generic** points of Teich_η . However, the set of I for which η is ample is **open and dense in Teich_η .**

Polarized moduli space

DEFINITION: Let $\Gamma_{I,\eta}$ be a subgroup of a mapping class group preserving η and a given component of Teich_η . Define **the polarized birational moduli space** as a quotient $(\text{Teich}_\eta / \sim) / \Gamma_{I,\eta}$.

THEOREM: (Anan'in-V.; this is the main result of this talk)

For each η , **the birational polarized moduli space is dense** in the birational moduli space Teich_b / Γ .

REMARK: For a K3 surface, **the moduli of quartic surfaces is dense in the moduli of all K3 surfaces**. The original proofs of many results on K3 surfaces were based on this observation.

Polarized moduli space and the 2-plane Grassmannian

The density theorem is implied by the following algebraic observation.

THEOREM: Let $V_{\mathbb{Z}} \subset V$ be an integer lattice in a vector space V equipped with an integral bilinear form of signature $(+++,-\dots-)$. Given a positive integer vector $\eta \in V_{\mathbb{Z}}$, denote by $\text{Gr}^{++}(\eta^{\perp}) \subset \text{Gr}^{++}(V)$ the space of all oriented, positive 2-planes orthogonal to η . Consider a finite index subgroup $G \subset \text{SO}(V_{\mathbb{Z}})$ acting on $\text{Gr}^{++}(V)$ in a natural way. **Then $G \cdot \text{Gr}^{++}(\eta^{\perp})$ is dense in $\text{Gr}^{++}(V)$.**

For our purposes, $V_{\mathbb{Z}} = (H^2(M, \mathbb{Z}), q)$ is a lattice of second cohomology of a hyperkähler manifold, and $\text{Gr}^{++}(\eta^{\perp}) = \text{Per}(\text{Teich}_{\eta})$, and $G = \Gamma_I$ the subgroup of the mapping class group preserving a connected component Teich_I .

Then, **density of $G \cdot \text{Gr}^{++}(\eta^{\perp})$ in $\text{Gr}^{++}(V)$ is equivalent to the density of the image of Teich_{η} in the birational moduli space Teich_b/Γ_I .**

REMARK: When $V_{\mathbb{Z}}$ is unimodular, the group $\text{SO}(V_{\mathbb{Z}})$ **acts transitively on the set of integer vectors of a given length in $V_{\mathbb{Z}}$** (“Eichler’s criterion”). Therefore, the orbit $G \cdot \eta$ is dense in $\mathbb{P}(V)$, hence $G \cdot \text{Gr}^{++}(\eta^{\perp})$ is dense in $\text{Gr}^{++}(V)$.

When $V_{\mathbb{Z}}$ is not unimodular, this is false.

Proof of the density theorem

Step 1: It suffices to prove it for V of signature $(+++ -)$. Since the rational subspaces are dense in $\text{Gr}_{++}(V)$, it suffices to show that any rational 2-plane $C \in \text{Gr}_{++}(V)$ belongs to the closure of $G \cdot \text{Gr}_{++}(\eta^\perp)$. Choose a rational space V_0 of signature $(3, 1)$ containing C and η , and notice that $G \cap \text{SO}(V_0)$ has finite index in $\text{SO}(V_0, \mathbb{Z})$.

From now on we assume that V has signature $(3, 1)$.

Step 2: Using the homeomorphism $\text{Gr}_{++}(V) \rightarrow \text{Gr}_{+-}(V)$, $G \mapsto G^\perp$ mapping the subspaces of signature $++$ to their orthogonal complements (of signature $+-$), we reformulate the density theorem as follows: **Every rational plane $C_0 \in \text{Gr}_{+-}(V)$ belongs to the closure of $G \cdot \{R \in \text{Gr}_{+-}(V) \mid R \ni l\}$.**

Step 3: Consider the space W (of signature $++ -$) generated by R and l . **The statement of Step 3 is implied by the following lemma.**

LEMMA: Let V be an \mathbb{R} -vector space equipped with a symmetric form of signature $++ -$, G a subgroup of finite index in $\text{O}(V_{\mathbb{Z}})$, where $V_{\mathbb{Z}}$ is a lattice in V , and $l \in V$ a positive vector. **Then $G \cdot \{C \in \text{Gr}_{+-}(V) \mid C \ni l\}$ is dense in $\text{Gr}_{+-}(V)$.**

The density lemma and the hyperbolic plane

Now we prove the lemma stated above:

LEMMA: Let V be an \mathbb{R} -vector space equipped with a symmetric form of signature $++-$, G a subgroup of finite index in $\mathcal{O}(V_{\mathbb{Z}})$, where $V_{\mathbb{Z}}$ is a lattice in V , and $l \in V$ a positive vector. **Then $G \cdot \{C \in \text{Gr}_{+-}(V) \mid C \ni l\}$ is dense in $\text{Gr}_{+-}(V)$.**

It is a statement about geometry of the hyperbolic plane \mathcal{H} .

REMARK: Identify \mathcal{H} with the projectivization of the negative cone in V . **Any geodesic on \mathcal{H} is obtained as a projectivization of a subspace $C \in \text{Gr}_{+-}(V)$.**

CLAIM: Let l be a positive vector in V , and C a $(+-)$ -plane. **Then $l \in C$ iff the geodesic $\mathbb{P}C$ is orthogonal to the geodesic $\mathbb{P}l^{\perp} \in \text{Gr}_{+-}(V)$.**

We reduced the lemma above to the following claim.

CLAIM: Let V be a vector space of signature $++-$, C, C' a pair of distinct geodesics, and G a subgroup of finite index in $SO(V_{\mathbb{Z}})$. Then for some $\gamma \in G$, C intersects $\gamma(C')$, and **the set of angles $\{\alpha = \angle(C, \gamma(C')) \mid \gamma \in G\}$ between those geodesics is dense in $[0, \pi]$.**

Open questions

QUESTION: This proof does not work when $q(\eta, \eta) \leq 0$. **Is the density theorem true for such η ?** It should be.

QUESTION: Let M_η be the polarized moduli space. It is quasiprojective and admits a complex analytic compactification (Baily-Borel compactification) **Is there a way to compactify the moduli space of all hyperkähler manifolds?** Should follow from Gromov's compactification.

QUESTION: Tosatti has shown that the **Gromov's limits have algebraic properties as long as the volume of the Kähler form stays bounded from below.** In particular, it is compatible with the Viehweg's universal family on the Baily-Borel compactification (proven in 1980-ies by Kobayashi, Todorov for K3). Is there a way to relate Gromov's limits of the set of **all** Ricci-flat metrics on M with algebraic geometry?

REMARK: Gromov's collapse has **obvious applications to the SYZ/abundance conjecture for hyperkähler manifolds.**