G-equivariant dual complex and embedded dd^c-lemma

Misha Verbitsky

UNICAMP, IMECC

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joint work with Nikon Kurnosov

Resolution of singularities

DEFINITION: Let $X \subset Y$ be a divisor in a smooth manifold. We say that X has **normal crossings** (or **NC**) if each of its components is smooth, and in a sufficiently small neighbourhood of every point of X the manifold Y has coordinate system such that X is identified with a union of coordinate hyperplanes. It has **simple normal crossings** (SNC) if it has normal crossings and for each collection of irreducible components of the divisor, their intersection is connected (possibly empty).

THEOREM: (Hironaka resolution)
Let X be a complex variety. Then there **a sequence of blow-ups with smooth centers** which results in a smooth resolution $\tilde{X} \stackrel{\pi}{\to} X$, and **the exceptional set of π is a SNC divisor.**

THEOREM: (Hironaka resolution of a pair)
Let X be a complex variety, $Y \subset X$ a subvariety. Then there **a sequence of blow-ups with smooth centers** which results in a smooth resolution $\tilde{X} \stackrel{\pi}{\to} X$, and **the exceptional set of π is a SNC divisor,** and **the preimage of Y is also an SNC divisor.**

Dual complex of a resolution

DEFINITION: (D. Stepanov, 2004)
Let $X \subset Y$ be an SNC divisor, and $X_1, ..., X_n$ its irreducible components. Its **dual complex** is a simplicial complex with k-simplices identified with non-empty intersections of $k+1$ divisors from $\{X_i\}$, and the boundary of a k-simplex associated with $C = X_{i_1} \cap ... \cap X_{i_{k+1}}$ is a $k-1$-simplex obtained as an intersection of all divisors in $X_{i_1}, ..., X_{i_{k+1}}$ except one.

THEOREM: (D. Stepanov)
Let $x \in X$ be an isolated singularity, and $\tilde X$ its smooth resolution, such that the preimage of x is an SNC divisor. Then **the homotopy type of its dual complex is independent from the choice of a resolution.**

Dual complex of a pair

Stepanov's theorem was generalized by Arapura-Bakhtary-Włodarczyk.

THEOREM: (D. Arapura, P. Bakhtary, J. Włodarczyk, 2011) Let $Y \subset X$ be a subvariety, and $\tilde{X} \xrightarrow{\pi} X$ the resolution of the pair $Y \subset X$. Let D be the exceptional set of π (it is an SNC divisor) and $\tilde{Y}$ the set of all components $D' \subset D$ such that $\pi(D') \subset Y$. **Then the homotopy type of the dual complex of $\tilde{Y}$ is independent from the choice of a resolution.**

COROLLARY: Let $\varphi : X_1 \to X_2$ be a holomorphic birational map, and $U_1 \subset X_1, U_2 \subset X_2$ open subsets such that the restriction $\varphi : U_1 \to U_2$ is proper. Assume that the complements $X_i \backslash U_i$ are SNC divisors. **Then their dual complexes are homotopy equivalent.**

In other words, **the homotopy type of an SNC divisor $D \subset X$ is uniquely determined by the class of the complement $X \backslash D$** up to proper bimeromorphic maps.

Kummer-type subvarieties

DEFINITION: We say that $Y \subset X$ is **of Kummer type** if $b_1(C(X,Y)) = 0$, where $C(X,Y)$ is the dual complex of this pair, and b_1 the rational Betti number.

THEOREM: Let $X \subset M$ be a complex subvariety of Kummer type, and G a finite group holomorphically acting on M and preserving X. **Then $X/G \subset M/G$ is Kummer type.**

Proof: Later today.

Weak factorization theorem

The following results are proven in D. Abramovich, K. Karu, K. Matsuki, J. Włodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc. **15** (2002), 531-572. and in K. Matsuki, *Lectures on factorization of birational maps*, arxiv:0002084.

THEOREM: (the weak factorization theorem)
Let $\Phi: X \dashrightarrow X'$ be a bimeromorphic map of compact complex manifolds, and $U \subset X$, $U' \subset X'$ the complements to the exceptional sets. Then Φ **can be decomposed into a sequence of blow-ups and blow-downs with smooth centers.**

We will need a version "with boundary".

THEOREM: (the weak factorization theorem)
Let $\Phi: X \dashrightarrow X'$ be a bimeromorphic map of compact complex manifolds, and $U \subset X$, $U' \subset X'$ the complements to the exceptional sets. Consider SNC divisors $D \subset X$, $D' \subset X'$, and let $D_1, ..., D_n$ and $D'_1, ..., D'_n$ be their irreducible components. Assume that Φ takes $X \backslash D_i$ to $X' \backslash D'_i$, and this bimeromorphic map is proper (that is, the closure of its graph in $X \backslash D_i \times X' \backslash D'_i$ projects properly to the components $X \backslash D_i$ and $X' \backslash D'_i$). Then Φ **can be decomposed into a sequence of blow-ups and blow-downs with smooth centers, and the exceptional sets of blow-ups have SNC with the proper preimages of $\bigcup D_i$.**

Proof: Theorem 7.7 in Arapura-Bakhtary-Włodarczyk. ■

Proof of Stepanov and Arapura-Bakhtary-Włodarczyk theorems

THEOREM: (D. Arapura, P. Bakhtary, J. Włodarczyk, 2011)
Let $Y \subset X$ be a subvariety, and $\tilde{X} \xrightarrow{\pi} X$ the resolution of the pair $Y \subset X$. Let D be the exceptional set of π (it is an SNC divisor) and $\tilde{Y}$ the set of all components $D' \subset D$ such that $\pi(D') \subset Y$. **Then the homotopy type of the dual complex of $\tilde{Y}$ is independent from the choice of a resolution.**

Step 1:
Suppose that $X_1 \to X$ and $X_2 \to X$ are two resolutions of a pair. Then the birational map $\Phi: X_1 \to X_2$ can be decomposed onto a a sequence of blow-ups and blow-downs with smooth centers. Therefore, to prove the homotopy invariance of a dual complex of a pair it **will suffice to show that a $C(\tilde{X}, \tilde{Y})$ is homotopy equivalent to $C(\tilde{X}_1, \tilde{Y}_1)$ when $\tilde{X}_1$ is obtained from $\tilde{X}$ by a blow-up, and the blow-up divisor is in SNC with the exceptional divisor of $\tilde{X}$.**

Step 2: Let $Z \subset \tilde{X}$ be the center of the blow-up $\tilde{X}_1 \to \tilde{X}$, and $A_1, ..., A_n$ the components of the SNC divisor $A = \tilde{Y}$. We need to consider several cases related to how Z intersects the components of A. **Case 0: $Z \notin \tilde{Y}$. In this case, $C(\tilde{X}, \tilde{Y}) = C(\tilde{X}_1, \tilde{Y}_1)$.**

Proof of Stepanov and Arapura-Bakhtary-Włodarczyk theorems

Step 3: Case 1: $Z = A_1 \cap A_2 \cap ... \cap A_r$. Let $A_Z \subset \tilde{X}_1$ be the corresponding blow-up divisor. In this case, $C(\tilde{X}_1, \tilde{Y}_1)$ contains an extra vertex, which corresponds to A_Z, and the corresponding divisor intersects the proper preimages $\tilde{A}_1, ..., \tilde{A}_r$. Any proper subset of divisors $\mathfrak{A} \subsetneq \{\tilde{A}_1, ..., \tilde{A}_r\}$ has a non-empty intersection outside of A_Z, hence intersects with A_Z, and $A_Z \cap \bigcap_{i=1}^r A_i = \emptyset$. This means that we removed from $C(\tilde{X}, \tilde{Y})$ an $r-1$-dimensional simplex which corresponds to $A_1 \cap A_2 \cap ... \cap A_r$ and glued instead a point A_Z and r $r-1$-dimensional simplices corresponding to $A_Z \cap \bigcap_{i=1..u-1, u+1, ..., r} A_i$. Therefore, **the simplicial complex $C(\tilde{X}_1, \tilde{Y}_1)$ is obtained from $C(\tilde{X}, \tilde{Y})$ by removing an $r-1$-simplex and gluing in the same simplex decomposed onto r $r-1$-simplices.** In other words, we replaced a simplex by its star decomposition. This implies that $C(\tilde{X}_1, \tilde{Y}_1)$ is homeomorphic to $C(\tilde{X}, \tilde{Y})$

Step 4: Case 2: the center Z is strictly contained in $A_1 \cap A_2 \cap ... \cap A_r$, but not contained in any smaller intersection. In this case, the blow-up divisor A_Z also intersects $\tilde{A}_1, ..., \tilde{A}_r$, and each subset of $\{\tilde{A}_i\}$ has non-empty intersection. However, the intersection $A_Z \cap \bigcap_i \tilde{A}_i$ is now non-empty. Therefore, **$C(\tilde{X}_1, \tilde{Y}_1)$ is obtained from $C(\tilde{X}, \tilde{Y})$ by gluing an r-dimensional simplex to an $r-1$-dimensional face associated with $A_1 \cap A_2 \cap ... \cap A_r$.** This simplicial complex is non-homeomorphic, but homotopy equivalent to $C(\tilde{X}, \tilde{Y})$. ■

G-Equivariant Hironaka resolution

THEOREM: Let G be a finite group acting on the pair $Y \subset X$ of compact complex varieties holomorphically. **Then there exists a G-equivariant SNC resolution of this pair, obtained by blow-ups with smooth G-invariant centers.** In other words, there exists a compact manifold $\tilde{X}$, equiped with an action of G, and a G-invariant holomorphic, bimeromorphic map $\pi : \tilde{X} \to X$, such that the exceptional set E of π is an SNC-divisor, the preimage of Y is an SNC-divisor, and the union $\pi^{-1}(Y) \cup E$ is also SNC.

Proof: This theorem was originally proved by Abramovich and Wang in Abramovich, Dan; Wang, Jianhua, *Equivariant resolution of singularities in characteristic 0*, Math. Res. Lett. 4 (1997), no. 2-3, 427-433. for algebraic varieties. In A, M. Bravo, S. Encinas, O. Villamayor U., *A Simplified Proof of Desingularization and Applications*, Rev. Mat. Iberoam. 21 (2005), no. 2, pp. 349-458, Bravo, Encinas, and Villamayor U. give a unified version of the proof, which is also valid for complex manifolds (Theorem 2.4). ■

G-Equivariant weak factorization theorem

THEOREM: (the weak factorization theorem)
Let X, X' be compact complex manifolds, equipped with a holomorphic action of a finite group G. Consider a G-equivariant bimeromorphic map $\Phi : X \dashrightarrow X'$, and let $U \subset X$, $U' \subset X'$ the complements to the exceptional sets. Then Φ **can be decomposed into a sequence of blow-ups and blow-downs with G-invariant smooth centers.**

Proof: K. Matsuki, Lectures on Factorization of Birational Maps, arXiv:math/0002084, Theorem 5-2-1. ■

Equivariant Arapura-Bakhtary-Włodarczyk theorem

THEOREM: Let G be a finite group acting on the pair $Y \subset X$ of compact complex varieties holomorphically. Consider G-equivariant SNC resolutions $\pi_1 : \tilde{X}_1 \to X$ and $\pi_2 : \tilde{X}_2 \to X$ of this pair, and let $C(\tilde{X}_i, \tilde{Y}_i)$ be the corresponding dual complex, considered as a simplicial space with an action of G by homeomorphisms. **Then $C(\tilde{X}_1, \tilde{Y}_1)$ is homotopy equivalent to $C(\tilde{X}_2, \tilde{Y}_2)$, and this homotopy can be made G-invariant.**

Proof: We use the G-equivariant weak factorization to decompose the bimeromorphic map $\Phi : \tilde{X}_1 \dashrightarrow \tilde{X}_2$ onto a composition of blow-ups and blow-downs with smooth G-invariant centers, and apply the same case-by-case argument as in Stepanov's and Arapura-Bakhtary-Włodarczyk's proof. On each step, we need to glue a G-invariant set of simplices, which gives us a homeomorphic dual complex (case 1) or a dual complex with a G-invariant set of simplices glued to the a G-invariant set of faces (case 2). ■

Discrepancy and dlt divisors

Reference to the following two slides: Tommaso de Fernex, János Kollár, Chenyang Xu, *The dual complex of singularities*, arXiv:1212.1675.

DEFINITION: Let $D \subset X$ be a $\mathbb{Q}$-divisor in a complex variety (that is, a formal sum of several divisors with rational coefficients (possibly negative). We say that (X, D) is **a pair** if all coefficients are $\leqslant 1$. We say that X is **$\mathbb{Q}$-Cartier** if a non-zero integer power of the canonical bundle has a section. Given a birational morphism $f : (Y, \Delta_Y) \to (X, \Delta_X)$, we define **the log pullback** of a pair (X, Δ_X) as a pair (Y, Δ_Y) defined by the formula $\Delta_Y + K_Y \sim_{\mathbb{Q}} f^*(\Delta_X + K_X)$, where $\sim_{\mathbb{Q}}$ denotes the rational equivalence of divisors, $A \sim_{\mathbb{Q}} B$ if $mA - mB$ is a divisor of a meromorphic function. Let $\Delta_Y := \sum a_i E_i$. **The discrepancy** of the divisor $E_i \subset Y$ is the number $-a_i$.

DEFINITION: A **log resolution** of a pair (X, Δ_X) is a birational morphism $f : (Y, \Delta_Y) \to (X, \Delta_X)$, where Y is smooth an Δ_Y SNC. A pair (X, Δ_X) is called **log canonical** if for any log-resolution $f : (Y, \Delta_Y) \to (X, \Delta_X)$, discrepancy of any divisor $E \subset Y$ is $\geqslant -1$. An **lc center** of (X, Δ_X) is $Z \subset X$ such that $f(E) = Z$ for some log-resolution and a divisor $E \subset Y$ if discrepancy -1.

Dlt divisors

DEFINITION: The **SNC locus** of a pair (X, Δ_X) is the set of all points $x \in X$ such that Δ_X is SNC in a neighbourhood of x. A pair (X, Δ_X) is **dlt** (divisorial log terminal) if for any log-resolutuion $f : (Y, \Delta_Y) \to (X, \Delta_X)$, any lc-center of f intersects the SNC locus of Δ_X.

REMARK: A dlt pair **is log-canonical and SNC outside of codimension 2.**

We treat the notion of dlt divisors as a black box. The use only the following two theorems, due to Kollár and de Fernex-Kollár-Xu.

DEFINITION: Let $D \subset X$ be a divisor. Assume that a non-empty intersection of k different irreducible components $D_1, ..., D_n$ of D has codimension $k+1$. Its **divisorial simplicial complex** is the following simplicial space. Its vertices are irreducible components of D. Its k-cells are irreducible components of the intersections of $(k-1)$ divisors in $\{D_i\}$, glued together the same way as above.

THEOREM: (Theorem 3, [dFKX])
Let $(Y, \Delta_Y) \to (X, \Delta_X)$ be an SNC resolution of a dlt pair, where all coefficients of Δ_X are 1. **Then the divisorial complex of Δ_Y is homotopy equivalent to the divisorial complex of Δ_X.**

Dual complex and finite quotients

THEOREM: Let G be a finite group acting on SNC pair (X, D) by automorphisms. **Then $(X/G, D/G)$ is a dlt pair.**

Proof: Kollar, Singularities of the Minimal Model Program. ■

COROLLARY: Let (X, S) be a pair, equipped with an action of a finite group G, and D its dual complex, obtained from a G-equivariant resolution and equipped with the natural action of G. **Then the Stepanov-ABW dual complex of the pair $(X/G, S/G)$ is homotopy equivalent to D/G.**

Proof: Let (X_1, S_1) be a G-equivariant SNC resolution of (X, S). Then $(X_1/G, D_1/G)$ is a dlt resolution of $(X/G, S/G)$, and Theorem 3 from dFKX implies that its Stepanov-ABW dual complex is homotopy equivalent to the divisorial simplicial complex of $(X_1/G, D_1/G)$. The latter complex is naturally identified with D/G. ■

COROLLARY: Let $X \subset M$ be a complex subvariety such that its dual complex satisfies $b_1 = 0$, and G a finite group holomorphically acting on M and preserving X. **Then the dual complex of $X/G \subset M/G$ satisfies $b_1 = 0$.**

Proof: $b_1(D) = 0 \Leftrightarrow b_1(D/G) = 0$ (Hatcher, Chapter 3, Proposition 3G.1). ■

dd^c-lemma

DEFINITION: Let M be a complex manifold, and $I : TM \longrightarrow TM$ its complex structure operator. **The twisted differential** of M is $IdI^{-1} : \Lambda^*(M) \longrightarrow \Lambda^{*+1}(M)$, where I acts on 1-forms as an operator dual to $I : TM \longrightarrow TM$, and on the rest of differential forms multiplicatively.

REMARK: Consider the Hodge decomposition of the de Rham differential, $d = \partial + \overline{\partial}$, where $\partial : \Lambda^{p,q}(M,I) \longrightarrow \Lambda^{p+1,q}(M,I)$ and $\overline{\partial} : \Lambda^{p,q}(M,I) \longrightarrow \Lambda^{p+1,q}(M,I)$. **Then $d = \operatorname{Re} \partial$ and $d^c = \operatorname{Im} \partial$.** Also, $dd^c = 2\sqrt{-1}\, \partial\overline{\partial}$.

THEOREM: (**dd^c-lemma**) Let η be a form on a compact Kähler manifold, satisfying one of the following conditions.
(1). η is an exact (p,q)-form. (2). η is d-exact, d^c-closed.
Then $\eta \in \operatorname{im} dd^c$.

dd^c-lemma and the dual complex

THEOREM 1: Let η be a real analytic (1,1)-form on M which is exact on a Fujiki class C subvariety $X \subset M$, which is normal and Kummer-type satisfies $b_1(S) = 0$, where S is the dual complex of the pair $X \subset M$. **Then there exists a subanalytic function f defined in a neighbourhood of X such that $\eta|{=}dd^c f|_X$.**

Proof. Step 1: Let $\tilde{X} \subset \tilde{M}$ be the resolution of the pair, and D_0 one of its components. Resolving again if necessary, we may assume that all components of $\tilde{X}$ are Kähler. **Then the pullback η_0 of η to D_0 is dd^c-exact,** because D_0 is smooth and Kähler, hence $\eta_0 = dd^c f$ for some function f.

Step 2: Let $\pi : \tilde{X}_0 \longrightarrow X$ be the projection, and $x \in X$ any point. Since $\ker dd^c$ is holomorphic plus antiholomorphic functions, on a compact complex manifold $\ker dd^c$ is constant functions. On each irreducible component of $\pi^{-1}(x)$, the function f satisfies $dd^c f = 0$, hence it is constant. Therefore, the function f such that $\eta_0 = dd^c f$ is uniquely, up to a constant, defined on each connected component D_i of $\tilde{X}$. To show that f is a pullback of a function on $\tilde{X}$, and hence on X, **we need only to chose these constants in such a way that $f|_{X\,i}$ agrees on all intersections $X_i \cap X_j$.**

dd^c-lemma and the dual complex (2)

Step 3: Choose a function f_i which satisfies $dd^c f_i = \eta|_{D_i}$ on each of these components. The difference $f_i\Big|_{D_i\cap D_j} - f_j\Big|_{D_i\cap D_j}$ is a constant function on each intersection $D_i \cap D_j$. Given a triple intersection $D_i \cap D_j \cap D_k \neq \emptyset$, we have $f_i\Big|_{D_i\cap D_j\cap D_k} - f_j\Big|_{D_i\cap D_j\cap D_k} + f_j\Big|_{D_i\cap D_j\cap D_k} - f_k\Big|_{D_i\cap D_j\cap D_k} + f_k\Big|_{D_i\cap D_j\cap D_k} - f_i\Big|_{D_i\cap D_j\cap D_k} = 0$, which gives a 1-cocycle on the dual complex S of the resolution. **To choose f_i which agree on the pairwise intersections, we need to show that this cocycle is exact,** which would follow if $H^1(S, \mathbb{R}) = 0$.

Step 4: Without restricting the generality, we can assume that η, and hence f_i, are real analytic on $\tilde{X}$; indeed, we can always choose η real analytic in its Bott-Chern cohomology class. To show that the pushforward of a function from $\tilde{X}$ to X is real analytic, we use normality of X. A pushforward of a real analytic function is subanalytic, if it is constant on fibers, and Every subanalytic function **can be extended to a small neighbourhood**. This gives a function f in a neighbourhood of X such that $dd^c f - \eta$ vanishes on X.

■