Eigenvalues of automorphisms of hyperkähler manifolds

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Eigenvalues of an automorphism of a hyperkähler manifold

THEOREM: (Bogomolov, Kamenova, Lu, V.)
Let (M, I) be a hyperkähler manifold, and f an automorphism of M. Assume that f acts on $H^2(M)$ with an eigenvalue $\alpha > 0$. **Then all eigenvalues of γ have absolute value which is a power of $u := \alpha^{1/2}$.** Moreover, **the maximal of these eigenvalues on even cohomology $H^{2d}(M)$ is equal to α^d (with eigenspace of dimension 1), and on odd cohomology $H^{2d+1}(M)$ it is strictly less than $\alpha^{\frac{2d+1}{2}}$.** Finally, let $H^p_k(M) \subset H^p(M)$ be the direct sum of all eigenspaces associated with eigenvalues α satisfying $|\alpha| = u^k$. **Then $dim H^p_k(M) = h^{k,p-k}(M)$.**

COROLLARY:

$$\lim_{n\to\infty} \frac{\log \operatorname{Tr}(f^n)\Big|_{H^*(M)}}{n} = \alpha.$$

In particular, **the number of k-periodic points grows as α^{nk}.**

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple**, or **maximal holonomy**, or **IHS** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

THEOREM: ("Bochner's vanishing")
Let M be a maximal holonomy hyperkähler manifold. **Then $H^{p,0} = 0$ for p odd, and $H^{p,0} = \mathbb{C}$ for p even.**

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.**

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - \frac{n-1}{2n} \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n}\right) \left(\int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: q has signature $(3, b_2 - 3)$. It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega \rangle$, where ω is a Kähler form.

K. Oguiso: Dynamical degree of an automorphism of a hyperkähler manifold

THEOREM: (K. Oguiso) Let $f: M \to M$ be an automorphism of a hyperkähler manifold with a real eigenvalue $\alpha > 1$ on $H^2(M)$. **Then** $h_{2d}(f) \geqslant \alpha^d$ **for all** $d \leqslant \dim_{\mathbb{H}}(M)$**.**

Proof: $H^{2d}(M)$ **contains the symmetric tensor product** $\operatorname{Sym}^d(H^2(M))$**.** ■

Problem: Not precise enough: **we don't get estimations of number of periodic points,** because we have no control over other eigenvalues.

Classification of automorphisms of hyperbolic space

REMARK: The group $O(m,n), m,n > 0$ has 4 connected components. We denote the connected component of 1 by $SO^+(m,n)$. We call a vector v **positive** if its square is positive.

DEFINITION: Let V be a vector space with quadratic form q of signature $(1,n)$, $\operatorname{Pos}(V) = \{x \in V \ \ | \ \ q(x,x) > 0\}$ its **positive cone**, and $\mathbb{P}^+V$ projectivization of $\operatorname{Pos}(V)$. Denote by g any $SO(V)$-invariant Riemannian structure on $\mathbb{P}^+V$. Then $(\mathbb{P}^+V, g)$ is called **hyperbolic space**, and the group $SO^+(V)$ **the group of oriented hyperbolic isometries**.

Theorem-definition: Let $n > 0$, and $\alpha \in SO^+(1,n)$ is an isometry acting on V. Then one and only one of these three cases occurs

(i) α has an eigenvector x with $q(x,x) > 0$ (α is **"elliptic isometry"**)

(ii) α has an eigenvector x with $q(x,x) = 0$ and a real eigenvalue λ_x satisfying $|\lambda_x| > 1$ (α is **"hyperbolic isometry"**)

(iii) α has a unique eigenvector x with $q(x,x) = 0$ (α is **"parabolic isometry"**).

REMARK: All eigenvalues of elliptic and parabolic isometries have absolute value 1. Hyperbolic and elliptic isometries are semisimple (that is, diagonalizable).

Automorphisms of hyperkahler manifolds

REMARK: Serge Cantat argues for a change of terminology to use **"loxodromic"** instead of "hyperbolic", and using "hyperbolic" for automorphisms which act trivially on a codimension 2 hyperspace.

DEFINITION: An automorphism of a hyperkähler manifold (M, I) is called **elliptic (parabolic, hyperbolic)** if it is elliptic (parabolic, hyperbolic) on $H_I^{1,1}(M, \mathbb{R})$.

THEOREM: (E. Amerik, V.)
Let M be a hyperkähler manifold, with $b_2(M) \geqslant 5$. Then M has a deformation admitting a hyperbolic automorphism.

THEOREM 1: (Bogomolov, Kamenova, Lu, V.)
Let M be a hyperkähler manifold, and $\gamma \in \operatorname{Aut}(H^*(M))$ an automorphism preserving the Hodge decomposition and acting on $H^{1,1}(M)$ hyperbolically. Denote by α the eigenvalue of γ on $H^2(M, \mathbb{R})$ with $|\alpha| > 1$. Replacing γ by γ^2 if necessary, we may assume that $\alpha > 1$. **Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$.** Moreover, **the eigenspace of eigenvalue $\alpha^{k/2}$ on $H^d(M)$ is isomorphic to $H^{\frac{(d+k)}{2}, \frac{(d-k)}{2}}(M)$.**

The proof of this result follows.

Hodge structures and automorphisms

REMARK: The Hodge decomposition **defines multiplicative action** of $U(1)$ on cohomology $H^*(M)$, with $t \in U(1) \subset \mathbb{C}$ acting on $H^{p,q}(M)$ as t^{p-q}.

THEOREM: (V., 1995) Let G be the group generated by $U(1)$-action for all complex structures on a hyperkähler manifold. **Then G is isomorphic to** $\operatorname{Spin}^+(H^2(M,\mathbb{R}),q)$ (with center acting trivially on even-dimensional forms and as -1 on odd-dimensional forms). Here Spin^+ denotes the connected component, and q is BBF form. ■

Theorem 2: The connected component of the group of automorphisms of $H^*(M)$ is mapped to G surjectively and with compact kernel.

Proof: $\operatorname{Aut}(H^*(M))$ is mapped to $SO(H^2(M,\mathbb{R}),q)$ by the restriction map; indeed, $\operatorname{Aut}(H^*(M))$ is compatible with the BBF form, as follows from the Fujiki theorem. It is surjective because $\operatorname{Aut}(H^*(M))$ contains the Hodge $U(1)$-action.

Finally, the kernel K of the map $\operatorname{Aut}(H^*(M)) \to G$ acts trivially on $H^2(M)$, hence commutes with the Lefschetz $SL(2)$-triples. However, the Hodge decomposition is expressed through the Lefschetz $SL(2)$-action by $\mathfrak{so}(1,4)$-theorem. Therefore, K also preserves the Hodge type. Therefore, K **preserves the Riemann-Hodge form, which is positive definite.** ■

$\operatorname{Aut}(H^*(M))$ is a direct product

Theorem 2: The connected component of the group of automorphisms of $H^*(M)$ is mapped to G surjectively and with compact kernel.

REMARK: By Theorem 2, the group $\operatorname{Aut}(H^*(M))$ is a semidirect product, $\operatorname{Aut}(H^*(M)) = G \ltimes K$. However, elements of K commute with elements of G, because they commute with the Hodge decomposition. **This gives $\operatorname{Aut}(H^*(M)) = K \times G$.**

COROLLARY: For each $f \in \operatorname{Aut}(H^*(M))$, **there exists an element $f' \in G = \operatorname{Spin}^+(H^2(M,\mathbb{R}), q)$ acting on $H^*(M)$ with eigenvalues of the same absolute value.**

Proof: Let $f = f'k$, where $f' \in G$, $k \in K$. Since k belongs to a compact group, all its eigenvalues have absolute value 1; since f' and k commute, eigenvalues of $f' = fk^{-1}$ are products of eigenvalues of f and eigenvalues of k. ■

COROLLARY: We obtain that **it suffices to prove Theorem 1 assuming that $\gamma \in G$.**

Eigenvalues of hyperbolic automorphisms

LEMMA: Let $\gamma \in SO(V^{1,n})$ be a hyperbolic automorphism of a vector space of signature $(1,n)$. **Then there exists $\gamma' \in SO(V^{1,n})$ with all eigenvalues equal 2 except 2 of them, commuting with γ and with $\gamma'\gamma^{-1}$ elliptic.**

Proof: Let α, α^{-1} be the eigenvalues of γ with absolute value $\neq 1$, and $X \subset V^{1,n}$ the corresponding 2-dimensional subspace. Then $X^\perp \subset V^{1,n}$ is a negative definite subspace preserved by γ.

Let γ' act as γ on X and as identity on $X^\perp$. Then $\gamma'\gamma^{-1}$ acts as isometry on $X^\perp$ and trivially on X, hence it has a positive eigenvector, and all its eigenvalues have absolute value 1. ■

REMARK: Since eigenvalues of γ and γ' on $H^*(M)$ have the same absolute values, **it suffices to prove Theorem 1 for γ equal to identity on a codimension 2 subspace.**

Eigenvalues of hyperbolic automorphisms

CLAIM: Let G be a group, and V its representation. **Then the eigenvalues of g and xgx^{-1} are equal for all $x, g \in G$.** ∎

PROPOSITION: Let (M, I) be a hyperkähler manifold, and $\gamma \in G = \mathrm{Spin}^+(H^2(M, \mathbb{R}), q)$ a hyperbolic isometry which acts as identity on a codimension 2 subspace in $H^2(M)$. Consider the one-parametric subgroup $H_\gamma := e^{\mathbb{C} \log \gamma}$ in the complexification $G_\mathbb{C}$ of G. Let W act on $H^{p,q}(M)$ as a multiplication by a scalar $\sqrt{-1}\,(p-q)$, and let $H = e^{\mathbb{C}W}$ be the corresponding one-parametric subgroups in $G_\mathbb{C}$. **Then H_γ and H are conjugate by some $h \in G_\mathbb{C}$.**

Proof: Both H and H_γ act on $H^2(M, C)$ with 2-dimensional eigenspaces X with eigenvalues λ, λ^{-1} and as identity on $X^\perp$. However, all such X are conjugate by some $h \in G_\mathbb{C}$. ∎

COROLLARY: The eigenvalue decomposition for γ acting on $H^*(M)$ is conjugate to the Hodge decomposition, and the eigenspaces with absolute value $\alpha^{k/2}$ under this conjugation correspond to $H^{p,q}(M)$ with $p - q = k$.

This finishes the proof of Theorem 1.

Topological entropy

DEFINITION: Let K be a metric space. A subset $S \subset K$ is called **ε-separated** if for all $x \neq y$ in S, $d(x,y) \geqslant \varepsilon$. Denote my $N(K,\varepsilon)$ the cardinality of a maximal ε-separated subset of $S \subset K$.

DEFINITION: Let (M,d) be a metric space, and $f: M \to M$ a self-map. Denote by d_n the metric $d_n(x,y) = \max_{k=0}^{n-1} d(f^k(x), f^k(y))$. The **topological entropy** of f is the number

$$h(f) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log N(M, d_n, \varepsilon)}{n}.$$

REMARK: Topological entropy counts the exponential growth of the number of ε-separated orbits.

Exercise: Assume that M is a compact metric space. **Prove that this number is independent from the choice of d.**

Gromov's theorem

DEFINITION: Let T be an automorphism of a manifold M, and consider the corresponding action on $H^d(M,\mathbb{R})$. **The d-th dynamical degree** is logarithm of the maximal absolute value of its eigenvalues.

THEOREM: (Gromov)
Let M be compact, Kähler, $f: M\to M$ its automorphism, $h_d(f)$ the d-th dynamical degree, and $h(f)$ topological entropy. **Then $f(h) = \max h_d(f)$.**

Proof: M. Gromov, On the entropy of holomorphic maps, `http://www.ihes.fr/~gromov/PDF/10%5B24%5D.pdf`, 1977.

S. Friedland, Entropy of algebraic maps, Proceedings of the Conference in Honor of Jean-Pierre Kahane, J. Fourier Anal. Appl. (1995), Special Issue, 215-228. `http://homepages.math.uic.edu/~friedlan/Dynalg.pdf` ■