Ergodic complex structures and Kobayashi metric

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Ergodic complex structures

DEFINITION: Let M be a smooth manifold. **A complex structure** on M is an endomorphism $I \in \operatorname{End} TM$, $I^2 = -\operatorname{Id}_{TM}$ such that the eigenspace bundles of I are **involutive**, that is, satisfy satisfy $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

REMARK: Let Comp be the space of such tensors equipped with a topology of convergence of all derivatives. **It is a Fréchet manifold**.

DEFINITION: The diffeomorphism group Diff is a Fréchet Lie group acting on Comp in a natural way. A complex structure is called **ergodic** if its Diff-orbit is dense in Comp.

REMARK: The "moduli space" of complex structures (if it exists) is identified with Comp / Diff; **existence of ergodic complex structures guarantees that the moduli space does not exist.**

THEOREM: Let M be a compact torus, $\dim_{\mathbb{C}} M \geqslant 2$, or a simple hyperkähler manifold. **A complex structure on M is ergodic if and only if $\operatorname{Pic}(M)$ is not of maximal rank.**

Teichmüller spaces

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let $\operatorname{Teich} := \operatorname{Comp}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on M** is a connected component of $\operatorname{Teich}/\Gamma$.

REMARK: This terminology is **standard for curves.**

Moduli spaces

DEFINITION: The quotient $\operatorname{Comp}/\operatorname{Diff} = \operatorname{Teich}/\Gamma$ is called **the moduli space** of complex structures. Typically, **it is very non-Hausdorff**. Comp corresponds bijectively to the set of isomorphism classes of complex structures.

REMARK: The moduli space exists, and is quasiprojective, for curves and manifolds with canonical polarization. The moduli space exists as a non-Hausdorff algebraic space when M is Kähler and $H^2(M) = H^{1,1}(M)$: Calabi-Yau manifolds, generalized Enriques manifolds, rational manifolds.

This talk is about an opposite situation, when Γ acts on Teich **ergodically.**

Holomorphically symplectic manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A compact hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

Computation of the mapping class group

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

THEOREM: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

THEOREM: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

REMARK: Sullivan's theorem implies that the mapping class group for $\dim_{\mathbb{C}} M \geqslant 3$ **is an arithmetic lattice**. **This is not true for curves.**

The period map

Remark: For any $J \in$ Teich, (M,J) is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.

Definition: Let $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

REMARK: P **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid \ q(l,l) = 0, q(l,\bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2-3,3)/SO(2) \times SO(b_2-3,1)$

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts) Two points $I, I' \in$ Teich **are non-separable if and only if there exists a bimeromorphism $(M, I) \longrightarrow (M, I')$ which is non-singular in codimension 2.**

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism, for each connected component of Teich_b.

DEFINITION: Let M be a hyperkaehler manifold, Teich_b its birational Teichmüller space, and Γ the mapping class group. The quotient $\operatorname{Teich}_b/\Gamma$ is called **the birational moduli space** of M.

THEOREM: Let (M, I) be a hyperkähler manifold, and W a connected component of its birational moduli space. **Then W is isomorphic to $\mathbb{P}\mathrm{er}/\Gamma$, where $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ and Γ is a finite index subgroup in $O(H^2(M, \mathbb{Z}), q)$**, called **the monodromy group**.

Ergodic complex structures

DEFINITION: Let (M, μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof. Step 1: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore the set Z_U of such orbits has measure 0.

Proof. Step 2: Choose a countable base $\{U_i\}$ of topology on M. Then the set of points in dense orbits is $M \backslash \bigcup_i Z_{U_i}$. ■

DEFINITION: Let M be a complex manifold, Teich its Techmüller space, and Γ the mapping group acting on Teich **An ergodic complex structure** is a complex structure with dense Γ-orbit.

CLAIM: Let (M, I) be a manifold with ergodic complex structure, and I' another complex structure. **Then there exists a sequence of diffeomorphisms ν_i such that $\nu_i^*(I)$ converges to I'.**

Ergodicity of the monodromy group action

DEFINITION: A lattice in a Lie group is a discrete subgroup $\Gamma \subset G$ such that G/Γ has finite volume with respect to Haar measure.

THEOREM: (Calvin C. Moore, 1966) Let Γ be a lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact subgroup. **Then the left action of Γ on G/H is ergodic.**

THEOREM: Let $\mathbb{P}\mathrm{er}$ be a component of a birational Teichmüller space, and Γ its monodromy group. Let $\mathbb{P}\mathrm{er}_e$ be a set of all points $L \subset \mathbb{P}\mathrm{er}$ such that the orbit $\Gamma \cdot L$ is dense (such points are called **ergodic**). **Then $Z := \mathbb{P}\mathrm{er} \setminus \mathbb{P}\mathrm{er}_e$ has measure 0.**

Proof. Step 1: Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. **Then Γ-action on G/H is ergodic,** by Moore's theorem.

Step 2: Ergodic orbits are dense, non-ergodic orbits have measure 0. ■

REMARK: Generic deformation of M has no rational curves, and no non-trivial birational models. Therefore, **outside of a measure zero subset,** $\mathrm{Teich} = \mathrm{Teich}_b$. This implies that **almost all complex structures on M are ergodic.**

Ratner's theorem

DEFINITION: Let G be a connected Lie group equipped with a Haar measure. **A lattice** $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, G/Γ has finite volume).

THEOREM: Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ an arithmetic lattice. Then **a closure of any H-orbit in G/Γ is an orbit of a closed, connected subgroup $S \subset G$, such that $S \cap \Gamma \subset S$ is a lattice.**

EXAMPLE: Let V be a real vector space with a non-degenerate bilinear symmetric form of signature $(3, k)$, $k > 0$, $G := SO^+(V)$ a connected component of the isometry group, $H \subset G$ a subgroup fixing a given positive 2-dimensional plane, $H \cong SO^+(1, k) \times SO(2)$, and $\Gamma \subset G$ an arithmetic lattice. Consider the quotient $\mathbb{P}\mathrm{er} := G/H$. Then

A). A point $J \in \mathbb{P}\mathrm{er}$ has dense Γ-orbit if and only if the orbit $H \cdot J$ in the quotient $\Gamma\backslash G$ is closed.

B). A closure of $H \cdot J$ in $\Gamma\backslash G$ is an orbit of a closed connected Lie group $S \supset H$.

Characterization of ergodic complex structures

CLAIM: Let $G = SO^+(3,k)$, and $H \cong SO^+(1,k) \times SO(2) \subset G$. Then **any closed connected Lie subgroup $S \subset G$ containing H coincides with G or with H.**

COROLLARY: Let $J \in \mathbb{P}\mathrm{er} = G/H$. Then **either J is ergodic, or its Γ-orbit is closed in $\mathbb{P}\mathrm{er}$.**

REMARK: By Ratner's theorem, in the latter case the H-orbit of J has finite volume in G/Γ. Therefore, **its intersection with Γ is a lattice in H.** This brings

COROLLARY: Let $J \in \mathbb{P}\mathrm{er}$ be a point, such that its Γ-orbit is closed in $\mathbb{P}\mathrm{er}$. Consider its stabilizer $\mathrm{St}(J) \cong H \subset G$. **Then $\mathrm{St}(J) \cap \Gamma$ is a lattice in $\mathrm{St}(J)$.**

COROLLARY: Let J be a non-ergodic complex structure on a hyperkähler manifold, and $W \subset H^2(M,\mathbb{R})$ be a plane generated by $\mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega$. **Then W is rational.**

REMARK: This can be used to show that **any hyperkähler manifold is Kobayashi non-hyperbolic.**

Kobayashi pseudometric

REMARK: The results further on are from a joint paper arXiv:1308.5667 by Ljudmila Kamenova, Steven Lu, Misha Verbitsky.

DEFINITION: Pseudometric on M is a function $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric: $d(x,y) = d(y,x)$ and satisfies the triangle inequality $d(x,y) + d(y,z) \geqslant d(x,z)$.

REMARK: Let $\mathfrak{D}$ be a set of pseudometrics. **Then $d_{\max}(x,y) := \sup_{d\in\mathfrak{D}} d(x,y)$ is also a pseudometric.**

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M is $d_{\max}$ for the set $\mathfrak{D}$ of all pseudometrics such that any holomorphic map from the Poincaré disk to M is distance-decreasing.

THEOREM: Let $\pi: \mathcal{M} \longrightarrow X$ be a smooth holomorphic family, which is trivialized as a smooth manifold: $\mathcal{M} = M \times X$, and d_x the Kobayashi metric on $\pi^{-1}(x)$. **Then $d_x(m,m')$ is upper continuous on x.** ■

COROLLARY: Denote the diameter of the Kobayashi pseudometric by $\operatorname{diam}(d_x) := \sup_{m,m'} d_x(m,m')$. **Then $\operatorname{diam}: X \longrightarrow \mathbb{R}^{\geqslant 0}$ is upper continuous.**

Vanishing of Kobayashi pseudometric

THEOREM: Let (M,I) be a complex manifold with vanishing Kobayashi pseudometric. Then **the Kobayashi pseudometric vanishes for all ergodic complex structures in the same deformation class.**

Proof: Let $\operatorname{diam}: \text{Comp} \longrightarrow \mathbb{R}^{\geqslant 0}$ map a complex structure J to the diameter of the Kobayashi pseudodistance on (M,J). Let J be an ergodic complex structure. The set of points $J' = \nu(J) \in \text{Comp}$ such that (M,J') is biholomorphic to (M,J) is dense, because J is ergodic. By upper semi-continuity, $0 = \operatorname{diam}(I) \geqslant \inf_{J'=\nu(J)} \operatorname{diam}(J)$. ■

EXAMPLE: Let M be a projective K3 surface. Then the Kobayashi metric on M vanishes. **Since all non-projective K3 are ergodic,** the Kobayashi metric vanishes on non-projective K3 surfaces as well.

THEOREM: Let M be a compact simple hyperkähler manifold. Assume that a deformation of M admits a holomorphic Lagrangian fibration and the Picard rank of M is not maximal. **Then the Kobayashi pseudometric on M vanishes.**

THEOREM: Let M be a Hilbert scheme of K3. **Then the Kobayashi pseudometric on M vanishes.**