Teichmuller spaces, ergodic theory and global Torelli theorem

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Teichmüller spaces

DEFINITION: Let M be a smooth manifold. **A complex structure** on M is an endomorphism $I \in \operatorname{End} TM$, $I^2 = -\operatorname{Id}_{TM}$ such that the eigenspace bundles of I are **involutive**, that is, satisfy satisfy $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

REMARK: Let Comp be the space of such tensors equipped with a topology of convergence of all derivatives. **It is a Fréchet manifold**.

REMARK: The diffeomorphism group Diff is a Fréchet Lie group acting on a Fréchet manifold Comp in a natural way.

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Let $\operatorname{Teich} := \operatorname{Comp} / \operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

REMARK: This terminology is **standard for curves.**

Moduli spaces

DEFINITION: The quotient $\operatorname{Comp}/\operatorname{Diff} = \operatorname{Teich}/\Gamma$ is called **the moduli space** of complex structures. It can be **very non-Hausdorff**. $\operatorname{Comp}/\operatorname{Diff}$ parametrizes the set of equivalence classes of complex structures.

REMARK: The moduli space exists, and is quasiprojective, for curves and projective manifolds with canonical or anticanonical polarization. **Its topology is extremely non-Hausdorff** for complex tori of higher dimension, hyperkähler manifolds, rational surfaces with $b_2 > 10$, and other varieties without a natural polarization.

REMARK: Teich **is a complex space, possibly non-Hausdorff** for a wide class of manifolds, including all Calabi-Yau (F. Catanese).

Holomorphically symplectic manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: Let $\omega_I, \omega_J, \omega_K$ be the Kähler symplectic forms associated with I, J, K. **A hyperkähler manifold is holomorphically symplectic:** $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I). **Converse is also true:**

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A compact hyperkähler manifold M is called **maximal holonomy manifold**, or **simple**, or **IHS** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: (Bogomolov, 1974) Any hyperkähler manifold admits a finite covering which is a product of a torus and several maximal holonomy (simple) hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

Computation of the mapping class group

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$,** and $c > 0$ a rational number.

DEFINITION: The form q is called **Bogomolov-Beauville-Fujiki form**. It has signature $(3, b_2 - 3)$.

THEOREM: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by $\Gamma_0 = \operatorname{Aut}(H^*(M,\mathbb{Z}), p_1, ..., p_n)$ the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the image of the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite index in Γ_0.**

THEOREM: (V., 1996, 2009) Let M be a simple hyperkähler manifold, and $\Gamma_0 = \operatorname{Aut}(H^*(M,\mathbb{Z}), p_1, ..., p_n)$. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}), q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}), q)$ **has finite kernel.**

REMARK: Sullivan's theorem implies that the mapping class group for a Kähler manifold M with $\dim_{\mathbb{C}} M \geqslant 3$, $\pi_1(M) = 0$ **is an arithmetic group**. Contrast that with the mapping class group of a Riemannian surface.

The period map

REMARK: To simplify the language, we redefine Teich and Comp for hyperkähler manifolds, admitting only complex structures of Kähler type. Since the Hodge numbers are constant in families of Kähler manifolds, **for any $J \in$ Teich, (M,J) is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.**

Definition: Let Per : Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map Per : Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

REMARK: **Per maps Teich into an open subset of a quadric,** defined by

$$\mathbb{P}\mathrm{er} := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l,\bar{l}) > 0\}.$$

It is called **the period space** of M.

REMARK: **$\mathbb{P}\mathrm{er} = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$.** Indeed, the group $SO(H^2(M,\mathbb{R}),q) = SO(b_2-3,3)$ acts transitively on $\mathbb{P}\mathrm{er}$, and $SO(2)\times SO(b_2-3,1)$ is a stabilizer of a point.

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts, 2001) Two points $I, I' \in \operatorname{Teich}$ **are non-separable if and only if there exists a bimeromorphism $(M, I) \longrightarrow (M, I')$ which is non-singular in codimension 2 and acts as identity on $H^2(M)$.**

REMARK: This is possible only if (M, I) and (M, I') contain a rational curve. **General hyperkähler manifold has no curves;** ones which have belong to a countable union of divisors in Teich.

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: (V., 2009) **The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism,** for each connected component of Teich_b.

The Hodge-theoretic Torelli theorem

DEFINITION: Let M be a hyperkaehler manifold. One says that **the Hodge-theoretic Torelli theorem holds for M** if the map

$$\text{Teich}/\Gamma_I \longrightarrow \mathbb{P}\text{er}/O^+(H^2(M,\mathbb{Z}),q),$$

is bijective, where $O^+(H^2(M,\mathbb{Z}),q)$ is a subgroup of $O(H^2(M,\mathbb{Z}),q)$ preserving orientation on positive 3-planes. Equivalently, **it is true if M is uniquely determined by its Hodge structure**.

REMARK: "Hodge-theoretic Torelli theorem" means that **the Hodge structure on $H^2(M)$ determines an isomorphism class of the manifold.**

REMARK: The Hodge-theoretic Torelli theorem **is true for K3 surfaces**. **It is false** for all other known examples of hyperkaehler manifolds.

Obstructions to Hodge-theoretic Torelli:
1. There exist bimeromorphic hyperkähler manifolds which are non-isomorphic, but have the same Hodge structures (Debarre, 1984).
2. **The covering** $\text{Teich}_b/\Gamma_I \longrightarrow \mathbb{P}\text{er}/O^+(H^2(M,\mathbb{Z}),q)$ **is non-trivial**, because $\Gamma_I \subsetneq O^+(H^2(M,\mathbb{Z}),q)$ (Namikawa, 2002).

Ergodic complex structures

DEFINITION: Let (M, μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof. Step 1: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore the set Z_U of such orbits has measure 0.

Proof. Step 2: Choose a countable base $\{U_i\}$ of topology on M. Then the set of points in dense orbits is $M \backslash \bigcup_i Z_{U_i}$. ■

DEFINITION: Let M be a complex manifold, Teich its Techmüller space, and Γ the mapping group acting on Teich. **An ergodic complex structure** is a complex structure with dense Γ-orbit.

CLAIM: Let (M, I) be a manifold with ergodic complex structure, and I' another complex structure. **Then there exists a sequence of diffeomorphisms ν_i such that $\nu_i^*(I)$ converges to I'.**

Ergodicity of the monodromy group action

DEFINITION: A lattice in a Lie group is a discrete subgroup $\Gamma \subset G$ such that G/Γ has finite volume with respect to Haar measure.

THEOREM: (Calvin C. Moore, 1966) Let Γ be a lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact subgroup. **Then the left action of Γ on G/H is ergodic.**

THEOREM: Let $\mathbb{P}\mathrm{er}$ be a component of a birational Teichmüller space, and Γ its monodromy group. Let $\mathbb{P}\mathrm{er}_e$ be a set of all orgodic $L \subset \mathbb{P}\mathrm{er}$. **Then $Z := \mathbb{P}\mathrm{er} \setminus \mathbb{P}\mathrm{er}_e$ has measure 0.**

Proof. Step 1: Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. **Then Γ-action on G/H is ergodic,** by Moore's theorem.

Step 2: Ergodic orbits are dense, non-ergodic orbits have measure 0. ■

REMARK: Generic deformation of M has no rational curves, and no non-trivial birational models. Therefore, **outside of a measure zero subset,** $\mathrm{Teich} = \mathrm{Teich}_b$**.** Then, Moore's theorem implies that **almost all complex structures on M are ergodic.**

Teichmüller space for a compact torus

DEFINITION: Let $\mathbb{Z}^{2n} \subset \mathbb{C}^n$ be a cocompact lattice. Then $\mathbb{C}^n/\mathbb{Z}^{2n}$ is a complex manifold, called **a (compact) complex torus**.

REMARK: The space of complex structures on R^{2n} is naturally identified with $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$.

CLAIM: Any connected component of the Teichmüller space for a compact torus is identified with $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$.

CLAIM: The action of $GL(2n,\mathbb{Z})$ on $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$ is ergodic.

Proof: Indeed, $SL(2n,\mathbb{Z})$ acts on $SL(2n,\mathbb{R})/SL(n,\mathbb{C})$ ergodically by Moore's theorem. ■

THEOREM: (V., 2013) Let $M = \mathbb{C}^n/\Lambda$ be a compact torus. **Then M is ergodic if and only if the lattice $\Lambda \cong \mathbb{Z}^{2n}$ is rational.**

Its proof uses Ratner theory.

Ratner's theorem

DEFINITION: Let G be a connected Lie group equipped with a Haar measure. **A lattice** $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, G/Γ has finite volume).

THEOREM: Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ an arithmetic lattice. Then **the closure of any Γ-orbit $\Gamma \cdot x$ in G/H is an orbit of a Lie subgroup $S \subset G$, such that $S \cap \Gamma^{x^{-1}}$ is a lattice in S.**

EXAMPLE: Let V be a real vector space with a non-degenerate bilinear symmetric form of signature $(3, k)$, $k > 0$, $G := SO^+(V)$ a connected component of the isometry group, $H \subset G$ a subgroup fixing a given positive 2-dimensional plane, $H \cong SO^+(1, k) \times SO(2)$, and $\Gamma \subset G$ an arithmetic lattice. Consider the quotient $\mathbb{P}\mathrm{er} := G/H$. **Then a closure of $\Gamma \cdot J$ in G/H is an orbit of a closed connected Lie group $S \supset H$.**

Characterization of ergodic complex structures

CLAIM: Let $G = SO^+(3,k)$, and $H \cong SO^+(1,k) \times SO(2) \subset G$. Then **any closed connected Lie subgroup $S \subset G$ containing H coincides with G or with H.**

COROLLARY: Let $J \in \mathbb{P}\mathrm{er} = G/H$. Then **either J is ergodic, or its Γ-orbit is closed in $\mathbb{P}\mathrm{er}$.**

REMARK: By Ratner's theorem, in the latter case the H-orbit of J has finite volume in G/Γ. Therefore, **its intersection with Γ is a lattice in H.** This brings

COROLLARY: Let $J \in \mathbb{P}\mathrm{er}$ be a point, such that its Γ-orbit is closed in $\mathbb{P}\mathrm{er}$. Consider its stabilizer $\mathrm{St}(J) \cong H \subset G$. **Then $\mathrm{St}(J) \cap \Gamma$ is a lattice in $\mathrm{St}(J)$.**

COROLLARY: Let J be a non-ergodic complex structure on a hyperkähler manifold, and $W \subset H^2(M,\mathbb{R})$ be a plane generated by $\mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega$. **Then W is rational.**

THEOREM: (V., 2013) Let M be a compact torus, $\dim_{\mathbb{C}} M \geqslant 2$, or a simple hyperkähler manifold. **A complex structure on M is ergodic if and only if $\mathrm{Pic}(M)$ is not of maximal rank.**

Kobayashi hyperbolic manifolds

DEFINITION: An entire curve is a non-constant holomorphic map $\mathbb{C} \longrightarrow M$.

DEFINITION: A compact complex manifold M is called **Kobayashi hyperbolic**, if there exist no entire curves $\mathbb{C} \longrightarrow M$.

THEOREM: (Brody, 1975)
Let I_i be a sequence of complex structures on M which are not hyperbolic, and I its limit. Then (M, I) is also not hyperbolic.

THEOREM: (V., 2013) **All hyperkähler manifolds are non-hyperbolic.**

REMARK: This result would follow if we produce an ergodic complex structure which is non-hyperbolic. Indeed, a closure of its orbit is the whole Teich, and a limit of non-hyperbolic complex structures is non-hyperbolic.

REMARK: For all **known** examples of hyperkähler manifolds, this result was already proven, due to L. Kamenova and M. V.

Twistor spaces and hyperkähler geometry

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

DEFINITION: **Induced complex structures** on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata).

Entire curves in twistor fibers

THEOREM: (F. Campana, 1992)
Let M be a hyperkähler manifold, and $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ its twistor projection. **Then there exists an entire curve in some fiber of π.**

CLAIM: There exists a twistor family which has only ergodic fibers.

Proof: There are only countably many complex structures which are not ergodic. ■

THEOREM: All hyperkähler manifolds are non-hyperbolic.

Proof: Let $\operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$ be a twistor family with all fibers ergodic. **By Campana's theorem, one of these fibers, denoted (M, I), is non-hyperbolic.** Since any complex structure $I' \in$ Teich lies in the closure of $\operatorname{Diff}(M) \cdot I$, all complex structures $I' \in$ Teich are non-hyperbolic. ■