Hyperkähler manifolds are non-hyperbolic

Misha Verbitsky

Moduli spaces of irreducible symplectic varieties,
cubics and Enriques surfaces
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Teichmüller spaces

**Definition:** Let $M$ be a compact complex manifold, and $\text{Diff}_0(M)$ a connected component of its diffeomorphism group (*the group of isotopies*). Denote by $\text{Comp}$ the space of complex structures on $M$, and let $\text{Teich} := \text{Comp} / \text{Diff}_0(M)$. We call it *the Teichmüller space*.

**Remark:** $\text{Teich}$ is *a finite-dimensional complex space* (Kodaira-Spencer-Kuranishi-Douady), but often *non-Hausdorff*.

**Definition:** Let $\text{Diff}_+(M)$ be the group of oriented diffeomorphisms of $M$. We call $\Gamma := \text{Diff}_+(M)/\text{Diff}_0(M)$ *the mapping class group*.

**REMARK:** Equivalence classes of complex structures on $M$ are *in bijective correspondence with elements of the quotient set* $\text{Teich} / \Gamma$.

To solve the moduli problem, one needs to describe $\text{Teich}$ and $\Gamma$. 

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**Holomorphically symplectic manifolds**

**DEFINITION:** A **hyperkähler structure** on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**REMARK:** A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1} \omega_K$ is a holomorphic symplectic form on $(M, I)$.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

**DEFINITION:** For the rest of this talk, a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.

**DEFINITION:** A compact hyperkähler manifold $M$ is called **simple**, or IHS, if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

**Bogomolov’s decomposition:** Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

**Further on, all hyperkähler manifolds are assumed to be simple/IHS.**
The Bogomolov-Beauville-Fujiki form

**THEOREM:** (Fujiki). Let \( \eta \in H^2(M) \), and \( \dim M = 2n \), where \( M \) is hyperkähler. Then \( \int_M \eta^{2n} = cq(\eta, \eta)^n \), for some primitive integer quadratic form \( q \) on \( H^2(M, \mathbb{Z}) \), and \( c > 0 \) an integer number.

**Definition:** This form is called *Bogomolov-Beauville-Fujiki form*. It is defined by the Fujiki’s relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville)

\[
\lambda q(\eta, \eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \frac{n-1}{n} \left( \int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n \right) \left( \int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1} \right)
\]

where \( \Omega \) is the holomorphic symplectic form, and \( \lambda > 0 \).
Computation of the mapping class group

**Theorem:** (Sullivan) Let $M$ be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geq 3$. Denote by $\Gamma_0$ the group of automorphisms of an algebra $H^*(M, \mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then the natural map $\text{Diff}^+(M)/\text{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in $\Gamma_0$.

**Theorem:** Let $M$ be a simple hyperkähler manifold, and $\Gamma_0$ as above. Then

(i) $\Gamma_0\big|_{H^2(M, \mathbb{Z})}$ is a finite index subgroup of $O(H^2(M, \mathbb{Z}), q)$.

(ii) The map $\Gamma_0 \longrightarrow O(H^2(M, \mathbb{Z}), q)$ has finite kernel.

**DEFINITION:** An arithmetic subgroup in a Lie group $G$ defined over rational numbers is a subgroup commensurable with the set $G_{\mathbb{Z}}$ of its integer points.

**COROLLARY:** The mapping class group of a hyperkähler manifold is an arithmetic group.

**REMARK:** The Teichmüller group (mapping class group of a Riemann surface) is not an arithmetic group.
The period map

**Remark:** For any $J \in \text{Teich}$, $(M,J)$ is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.

**Definition:** Let $P : \text{Teich} \longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map $J$ to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map $P : \text{Teich} \longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called the period map.

**Remark:** $P$ maps Teich into an open subset of a quadric, defined by

$$\text{Per} := \{ l \in \mathbb{P}H^2(M,\mathbb{C}) \mid q(l,l) = 0, q(l,l) > 0 \}.$$  

It is called the period space of $M$.
**Period space as a Grassmannian of positive 2-planes**

**PROPOSITION:** The period space

$$\text{Per} := \{ l \in \mathbb{P} H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0 \}.$$  

is identified with $SO(b_2-3, 3)/SO(2) \times SO(b_2-3, 1)$, which is a Grassmannian of positive oriented 2-planes in $H^2(M, \mathbb{R})$.

**Proof. Step 1:** Given $l \in \mathbb{P} H^2(M, \mathbb{C})$, the space generated by $\text{Im} l, \text{Re} l$ is 2-dimensional, because $q(l, l) = 0, q(l, \bar{l})$ implies that $l \cap H^2(M, \mathbb{R}) = 0$.

**Step 2:** This 2-dimensional plane is positive, because $q(\text{Re} l, \text{Re} l) = q(l + \bar{l}, l + \bar{l}) = 2q(l, \bar{l}) > 0$.

**Step 3:** Conversely, for any 2-dimensional positive plane $V \in H^2(M, \mathbb{R})$, the quadric $\{ l \in V \otimes_{\mathbb{R}} \mathbb{C} \mid q(l, l) = 0 \}$ consists of two lines; a choice of a line is determined by orientation. ■
**Birational Teichmüller moduli space**

**DEFINITION:** Let $M$ be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

**THEOREM:** (Huybrechts) Two points $I, I' \in \text{Teich}$ are non-separable if and only if there exists a bimeromorphism $(M, I) \to (M, I')$ which is non-singular in codimension 2.

**DEFINITION:** The space $\text{Teich}_b := \text{Teich} / \sim$ is called the **birational Teichmüller space** of $M$.

**THEOREM:** The period map $\text{Teich}_b \xrightarrow{\text{Per}} \text{Per}$ is an isomorphism, for each connected component of $\text{Teich}_b$.

**DEFINITION:** Let $M$ be a hyperkaehler manifold, $\text{Teich}_b$ its birational Teichmüller space, and $\Gamma$ the mapping class group. The quotient $\text{Teich}_b / \Gamma$ is called the **birational moduli space** of $M$. 
Monodromy group and the birational moduli space

**THEOREM:** Let \((M, I)\) be a hyperkähler manifold, and \(W\) the set of birational classes of complex structures (holomorphically symplectic, Kähler type) in a connected component of its deformation space. Then \(W\) is isomorphic to \(\mathbb{P}_{\text{er}}/\Gamma_I\), where \(\mathbb{P}_{\text{er}} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)\) and \(\Gamma_I\) is an arithmetic subgroup in \(O(H^2(M, \mathbb{R}), q)\), called the monodromy group.

**REMARK:** \(\Gamma_I\) is a group generated by monodromy of the Gauss-Manin local system on \(H^2(M)\) (Markman).

**A CAUTION:** Usually “the global Torelli theorem” is understood as a theorem about Hodge structures. For K3 surfaces, the Hodge structure on \(H^2(M, \mathbb{Z})\) determines the complex structure. For \(\dim_{\mathbb{C}} M > 2\), it is false.

**REMARK:** Further on, I shall freely identify \(\mathbb{P}_{\text{er}}\) and \(\text{Teich}_b\).
Ergodic complex structures

**Definition:** Let \((M, \mu)\) be a space with measure, and \(G\) a group acting on \(M\) preserving measure. This action is **ergodic** if all \(G\)-invariant measurable subsets \(M' \subset M\) satisfy \(\mu(M') = 0\) or \(\mu(M \setminus M') = 0\).

**Claim:** Let \(M\) be a manifold, \(\mu\) a Lebesgue measure, and \(G\) a group acting on \((M, \mu)\) ergodically. Then the set of non-dense orbits has measure 0.

**Proof:** Consider a non-empty open subset \(U \subset M\). Then \(\mu(U) > 0\), hence \(M' := G \cdot U\) satisfies \(\mu(M \setminus M') = 0\). For any orbit \(G \cdot x\) not intersecting \(U\), \(x \in M \setminus M'\). Therefore the set of such orbits has measure 0. 

**Definition:** Let \(M\) be a complex manifold, \(\text{Teich}\) its \(\text{Teichm"uller}\) space, and \(\Gamma\) the mapping group acting on \(\text{Teich}\). An **ergodic complex structure** is a complex structure with dense \(\Gamma\)-orbit.

**Claim:** Let \((M, I)\) be a manifold with ergodic complex structure, and \(I'\) another complex structure. Then there exists a sequence of diffeomorphisms \(\nu_i\) such that \(\nu_i^*(I)\) converges to \(I'\).
Ergodic complex structures

M. Verbitsky

Ergodicity of the monodromy group action

The moduli space $\mathbb{P}e r/\Gamma_I$ is extremely non-Hausdorff.

**THEOREM:** (Calvin C. Moore, 1966) Let $\Gamma$ be an arithmetic subgroup in a non-compact simple Lie group $G$ with finite center, and $H \subset G$ a non-compact subgroup. Then the left action of $\Gamma$ on $G/H$ is ergodic.

**THEOREM:** Let $\mathbb{P}e r$ be a component of a birational Teichmüller space, and $\Gamma$ its monodromy group. Let $\mathbb{P}e r_e$ be a set of all points $L \subset \mathbb{P}e r$ such that the orbit $\Gamma \cdot L$ is dense (such points are called ergodic). Then $Z := \mathbb{P}e r \setminus \mathbb{P}e r_e$ has measure 0.

**Proof. Step 1:** Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. Then $\Gamma$-action on $G/H$ is ergodic, by Moore’s theorem.

**Step 2:** Ergodic orbits are dense, non-ergodic orbits have measure 0. $\blacksquare$

**REMARK:** This implies that “almost all” $\Gamma$-orbits in $G/H$ are dense.

**REMARK:** Generic deformation of $M$ has no rational curves, and no non-trivial birational models. Therefore, outside of a measure zero subset, $\text{Teich} = \text{Teich}_{b}$. This implies that almost all complex structures on $M$ are ergodic.
Ratner’s theorem

**DEFINITION:** Let $G$ be a connected Lie group equipped with a Haar measure. A lattice $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, $G/\Gamma$ has finite volume).

**EXAMPLE:** By Borel and Harish-Chandra theorem, any integer lattice in a simple Lie group has finite covolume.

**THEOREM:** Let $H \subset G$ be a Lie subgroup generated by unipotents, and $\Gamma \subset G$ a lattice. Then a closure of any $H$-orbit in $G/\Gamma$ is an orbit of a closed, connected subgroup $S \subset G$, such that $S \cap \Gamma \subset S$ is a lattice.

**REMARK:** Let $x \in G/H$ be a point in a homogeneous space, and $\Gamma \cdot x$ its $\Gamma$-orbit, where $\Gamma$ is an arithmetic lattice. Then its closure is an orbit of a group $S$ containing stabilizer of $x$. Moreover, $S$ is a smallest group defined over rationals and stabilizing $x$. 

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Characterization of ergodic complex structures

**CLAIM:** Let \( G = SO^+(3,k) \), and \( H \cong SO^+(1,k) \times SO(2) \subset G \). Then any closed connected Lie subgroup \( S \subset G \) containing \( H \) coincides with \( G \) or with \( H \).

**COROLLARY:** Let \( J \in \text{Per} = G/H \). Then either \( J \) is ergodic, or its \( \Gamma \)-orbit is closed in \( \text{Per} \).

**REMARK:** By Ratner’s theorem, in the latter case the \( H \)-orbit of \( J \) has finite volume in \( G/\Gamma \). Therefore, its intersection with \( \Gamma \) is a lattice in \( H \). This brings

**COROLLARY:** Let \( J \in \text{Per} \) be a point such that its \( \Gamma \)-orbit is closed in \( \text{Per} \). Consider its stabilizer \( \text{St}(J) \cong H \subset G \). Then \( \text{St}(J) \cap \Gamma \) is a lattice in \( \text{St}(J) \).

**COROLLARY:** Let \( J \) be a non-ergodic complex structure on a hyperkähler manifold, and \( W \subset H^2(M,\mathbb{R}) \) be a plane generated by \( \text{Re}\Omega, \text{Im}\Omega \). Then \( W \) is rational. Equivalently, this means that \( \text{Pic}(M) \) has maximal possible dimension.

**REMARK:** This can be used to show that any hyperkähler manifold is Kobayashi non-hyperbolic.
Kobayashi hyperbolic manifolds

**DEFINITION:** An entire curve is a non-constant map $\mathbb{C} \to M$.

**DEFINITION:** A compact complex manifold $M$ is called **Kobayashi hyperbolic**, if there exist no entire curves $\mathbb{C} \to M$.

**THEOREM:** (Brody, 1975)
Let $I_i$ be a sequence of complex structures on $M$ which are not hyperbolic, and $I$ its limit. Then $(M, I)$ is also not hyperbolic.

**THEOREM:** All hyperkähler manifolds are non-hyperbolic.

**REMARK:** This conjecture would follow if we produce an ergodic complex structure which is non-hyperbolic. Indeed, a closure of its orbit is the whole Teich, and a limit of non-hyperbolic complex structures is non-hyperbolic.
**Twistor spaces and hyperkähler geometry**

**DEFINITION:** A hyperkähler structure on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**DEFINITION:** Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{ L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1 \}$.

**DEFINITION:** A twistor space $\text{Tw}(M)$ of a hyperkähler manifold is a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$. More formally:

Let $\text{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on $M$ induced by $J \in S^2 \subset \mathbb{H}$. Let $I_J$ denote the complex structure on $S^2 \cong \mathbb{C}P^1$.

The operator $I_{\text{Tw}} = I_m \oplus I_J : T_x \text{Tw}(M) \to T_x \text{Tw}(M)$ satisfies $I_{\text{Tw}}^2 = -\text{Id}$. It defines an almost complex structure on $\text{Tw}(M)$. This almost complex structure is known to be integrable (Obata).
Entire curves in twistor fibers

**THEOREM: (F. Campana, 1992)**
Let $M$ be a hyperkähler manifold, and $\text{Tw}(M) \xrightarrow{\pi} \mathbb{CP}^1$ its twistor projection. Then there exists an entire curve in some fiber of $\pi$.

**CLAIM:** There exists a twistor family which has only ergodic fibers.

**Proof:** There are only countably many complex structures which are not ergodic. ■

**THEOREM:** All hyperkähler manifolds are non-hyperbolic.

**Proof:** Let $\text{Tw}(M) \rightarrow \mathbb{CP}^1$ be a twistor family with all fibers ergodic. By Campana’s theorem, one of these fibers, denoted $(M,I)$, is non-hyperbolic. Since any complex structure $I' \in \text{Teich}$ lies in the closure of $\text{Diff}(M) \cdot I$, all complex structures $I' \in \text{Teich}$ are non-hyperbolic. ■