Hyperkähler manifolds are non-hyperbolic

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Moduli spaces of irreducible symplectic varieties,
cubics and Enriques surfaces
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Teichmüller spaces

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let $\operatorname{Teich} := \operatorname{Comp}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**.

REMARK: Equivalence classes of complex structures on M are **in bijective correspondence with elements of the quotient set** $\operatorname{Teich}/\Gamma$**.**

To solve the moduli problem, one needs to describe Teich **and** Γ**.**

Holomorphically symplectic manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A compact hyperkähler manifold M is called **simple**, or IHS, if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple/IHS.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\begin{aligned}\lambda q(\eta,\eta) = & \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ & - \frac{n-1}{n} \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n \right) \left(\int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1} \right)\end{aligned}$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb C} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb Z)$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb Z)}$ **is a finite index subgroup of $O(H^2(M,\mathbb Z),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb Z),q)$ **has finite kernel.**

DEFINITION: An arithmetic subgroup in a Lie group G defined over rational numbers is a subgroup commensurable with the set $G_{\mathbb Z}$ of its integer points.

COROLLARY: The mapping class group of a hyperkähler manifold **is an arithmetic group**.

REMARK: The Teichmüller group (mapping class group of a Riemann surface) is **not** an arithmetic group.

The period map

Remark: For any $J \in$ Teich, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $P :$ Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $P :$ Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: P **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

It is called **the period space** of M.

Period space as a Grassmannian of positive 2-planes

PROPOSITION: The period space

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l, \bar{l}) > 0\}.$$

is identified with $SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$**,** which is a Grassmannian of positive oriented 2-planes in $H^2(M, \mathbb{R})$.

Proof. Step 1: Given $l \in \mathbb{P}H^2(M, \mathbb{C})$, **the space generated by** $\operatorname{Im} l, \operatorname{Re} l$ **is 2-dimensional,** because $q(l,l) = 0, q(l, \bar{l})$ implies that $l \cap H^2(M, \mathbb{R}) = 0$.

Step 2: This 2-dimensional plane is positive, because $q(\operatorname{Re} l, \operatorname{Re} l) = q(l + \bar{l}, l + \bar{l}) = 2q(l, \bar{l}) > 0$.

Step 3: Conversely, for any 2-dimensional positive plane $V \in H^2(M, \mathbb{R})$, **the quadric** $\{l \in V \otimes_{\mathbb{R}} \mathbb{C} \ \ | \ \ q(l,l) = 0\}$ **consists of two lines;** a choice of a line is determined by orientation. ■

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts) Two points $I, I' \in \operatorname{Teich}$ **are non-separable if and only if there exists a bimeromorphism** $(M, I) \longrightarrow (M, I')$ **which is non-singular in codimension 2.**

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ **is an isomorphism,** for each connected component of Teich_b.

DEFINITION: Let M be a hyperkaehler manifold, Teich_b its birational Teichmüller space, and Γ the mapping class group. The quotient $\operatorname{Teich}_b/\Gamma$ is called **the birational moduli space** of M.

Monodromy group and the birational moduli space

THEOREM: Let (M, I) be a hyperkähler manifold, and W the set of birational classes of complex structures (holomorphically symplectic, Kähler type) in a connected component of its deformation space. **Then W is isomorphic to $\mathbb{P}\text{er}/\Gamma_I$, where $\mathbb{P}\text{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ and Γ_I is an arithmetic subgroup in $O(H^2(M, \mathbb{R}), q)$**, called **the monodromy group**.

REMARK: Γ_I is a group generated by monodromy of the Gauss-Manin local system on $H^2(M)$ (Markman).

A CAUTION: Usually "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M, \mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, **it is false**.

REMARK: Further on, **I shall freely identify $\mathbb{P}\text{er}$ and Teich_b.**

Ergodic complex structures

DEFINITION: Let (M,μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on (M,μ) ergodically. **Then the set of non-dense orbits has measure 0.**

Proof: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore the set of such orbits has measure 0. ■

DEFINITION: Let M be a complex manifold, Teich its Techmüller space, and Γ the mapping group acting on Teich **An ergodic complex structure** is a complex structure with dense Γ-orbit.

CLAIM: Let (M,I) be a manifold with ergodic complex structure, and I' another complex structure. **Then there exists a sequence of diffeomorphisms ν_i such that $\nu_i^*(I)$ converges to I'.**

Ergodicity of the monodromy group action

The moduli space $\mathbb{P}\mathrm{er}/\Gamma_I$ **is extremely non-Hausdorff**.

THEOREM: (Calvin C. Moore, 1966) Let Γ be an arithmetic subgroup in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact subgroup. **Then the left action of Γ on G/H is ergodic.**

THEOREM: Let $\mathbb{P}\mathrm{er}$ be a component of a birational Teichmüller space, and Γ its monodromy group. Let $\mathbb{P}\mathrm{er}_e$ be a set of all points $L \subset \mathbb{P}\mathrm{er}$ such that the orbit $\Gamma \cdot L$ is dense (such points are called **ergodic**). **Then $Z := \mathbb{P}\mathrm{er} \setminus \mathbb{P}\mathrm{er}_e$ has measure 0.**

Proof. Step 1: Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. **Then Γ-action on G/H is ergodic,** by Moore's theorem.

Step 2: Ergodic orbits are dense, non-ergodic orbits have measure 0. ■

REMARK: This implies that **"almost all" Γ-orbits in G/H are dense**.

REMARK: Generic deformation of M has no rational curves, and no non-trivial birational models. Therefore, **outside of a measure zero subset,** $\mathrm{Teich} = \mathrm{Teich}_b$. This implies that **almost all complex structures on M are ergodic.**

Ratner's theorem

DEFINITION: Let G be a connected Lie group equipped with a Haar measure. **A lattice** $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, G/Γ has finite volume).

EXAMPLE: By Borel and Harish-Chandra theorem, **any integer lattice in a simple Lie group has finite covolume.**

THEOREM: Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ a lattice. Then **a closure of any H-orbit in G/Γ is an orbit of a closed, connected subgroup $S \subset G$, such that $S \cap \Gamma \subset S$ is a lattice.**

REMARK: Let $x \in G/H$ be a point in a homogeneous space, and $\Gamma \cdot x$ its Γ-orbit, where Γ is an arithmetic lattice. Then its closure is an orbit of a group S containing stabilizer of x. Moreover, **S is a smallest group defined over rationals and stabilizing x.**

Characterization of ergodic complex structures

CLAIM: Let $G = SO^+(3,k)$, and $H \cong SO^+(1,k) \times SO(2) \subset G$. Then **any closed connected Lie subgroup $S \subset G$ containing H coincides with G or with H.**

COROLLARY: Let $J \in \mathbb{P}\mathrm{er} = G/H$. Then **either J is ergodic, or its Γ-orbit is closed in $\mathbb{P}\mathrm{er}$.**

REMARK: By Ratner's theorem, in the latter case the H-orbit of J has finite volume in G/Γ. Therefore, **its intersection with Γ is a lattice in H.** This brings

COROLLARY: Let $J \in \mathbb{P}\mathrm{er}$ be a point such that its Γ-orbit is closed in $\mathbb{P}\mathrm{er}$. Consider its stabilizer $\mathrm{St}(J) \cong H \subset G$. **Then $\mathrm{St}(J) \cap \Gamma$ is a lattice in $\mathrm{St}(J)$.**

COROLLARY: Let J be a non-ergodic complex structure on a hyperkähler manifold, and $W \subset H^2(M, \mathbb{R})$ be a plane generated by $\mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega$. **Then W is rational.** Equivalently, this means that $Pic(M)$ has maximal possible dimension.

REMARK: This can be used to show that **any hyperkähler manifold is Kobayashi non-hyperbolic.**

Kobayashi hyperbolic manifolds

DEFINITION: An entire curve is a non-constant map $\mathbb{C} \longrightarrow M$.

DEFINITION: A compact complex manifold M is called **Kobayashi hyperbolic**, if there exist no entire curves $\mathbb{C} \longrightarrow M$.

THEOREM: (Brody, 1975)
Let I_i be a sequence of complex structures on M which are not hyperbolic, and I its limit. Then (M, I) is also not hyperbolic.

THEOREM: All hyperkähler manifolds are non-hyperbolic.

REMARK: This conjecture would follow if we produce an ergodic complex structure which is non-hyperbolic. Indeed, a closure of its orbit is the whole Teich, and a limit of non-hyperbolic complex structures is non-hyperbolic.

Twistor spaces and hyperkähler geometry

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

DEFINITION: **Induced complex structures** on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$

DEFINITION: A **twistor space** $\mathrm{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\mathrm{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\mathrm{Tw}} = I_m \oplus I_J : T_x \mathrm{Tw}(M) \to T_x \mathrm{Tw}(M)$ satisfies $I_{\mathrm{Tw}}^2 = -\mathrm{Id}$. **It defines an almost complex structure on $\mathrm{Tw}(M)$.** This almost complex structure is known to be integrable (Obata).

Entire curves in twistor fibers

THEOREM: (F. Campana, 1992)
Let M be a hyperkähler manifold, and $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ its twistor projection. **Then there exists an entire curve in some fiber of π.**

CLAIM: There exists a twistor family which has only ergodic fibers.

Proof: There are only countably many complex structures which are not ergodic. ■

THEOREM: All hyperkähler manifolds are non-hyperbolic.

Proof: Let $\operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$ be a twistor family with all fibers ergodic. **By Campana's theorem, one of these fibers, denoted (M, I), is non-hyperbolic.** Since any complex structure $I' \in$ Teich lies in the closure of $\operatorname{Diff}(M) \cdot I$, all complex structures $I' \in$ Teich are non-hyperbolic. ■