Moduli spaces and the mapping class group

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Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

DEFINITION: The space of almost complex structures is an infinite-dimensional Fréchet manifold X_M of all tensors $I^2 = -\operatorname{Id}_{TM}$, equipped with the natural Fréchet topology.

CLAIM: The space Comp of integrable almost complex structures **is a submanifold in X_M** (also infinite-dimensional).

Teichmüller space

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let Teich := Comp / $\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

REMARK: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

DEFINITION: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **moduli space of complex structures on M** is a connected component of Teich $/\Gamma$.

REMARK: This terminology is **standard for curves.**

REMARK: The topology of the "moduli space" Teich $/\Gamma$ is often bizzarre. However, **its points are in bijective correspondence with equivalence classes of complex structures.**

REMARK: To describe the moduli of complex structures:
* **we need to describe** Teich (which is usually an OK complex space)
* **and the maping class group** Γ (for $\dim_{\mathbb{C}} M > 2$, it is an arithmetic group, described explicitly in terms of cohomology).

Kähler manifolds

DEFINITION: An Riemannian metric g on a complex manifold (M, I) is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. **Fubini-Study form on $T_x\mathbb{C}P^n = \mathbb{C}^n$ is $U(n)$-invariant, hence unique up to a constant.**

Kähler manifolds II.

Claim: Fubini-Study form is Kähler. Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish, because $-\operatorname{Id} \in U(n)$

REMARK: The same argument works for all symmetric spaces.

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler. Indeed, a restriction of a closed form is again closed.

DEFINITION: The cohomology class of the Kähler form is called **the Kähler class** of a manifold.

Hodge theory for Kähler manifolds (first cohomology):
Let M be a compact Kähler manifold, and $\theta \in \Omega^1(M)$ a holomorphic differential. Then θ is closed, and its cohomology class is non-zero. This gives an injective map $\Psi: H^0(\Omega^1 M) \hookrightarrow H^1(M, \mathbb{C})$ from the space of holomorphic differentials to cohomology. Moreover, **any $\alpha \in H^1(M, \mathbb{C})$ can be decomposed as $\alpha = \alpha^{1,0} + \alpha^{0,1}$, with $\alpha^{1,0} \in \operatorname{im} \Psi$ and $\overline{\alpha^{0,1}} \in \operatorname{im} \Psi$ represented by holomorphic differentials.**

DEFINITION: The space $\operatorname{im} \Psi$ is denoted $H^{1,0}(M)$.

Teichmüller space for a compact torus

DEFINITION: Let $\mathbb{Z}^{2n} \subset \mathbb{C}^n$ be a cocompact lattice. Then $\mathbb{C}^n/\mathbb{Z}^{2n}$ is a complex manifold, called **a (compact) complex torus**.

REMARK: The space of complex structures on R^{2n} is naturally identified with $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$.

THEOREM: Any connected component of the Teichmüller space for a compact torus is identified with $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$.

Proof: Let the **period map** put (M,I) to $H^{1,0}(M) \subset H^1(M,\mathbb{C})$, considered as a point on $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$. **Since $M = H^{1,0}(M)/H^1(M,\mathbb{Z})$, this map is invertible.** ■

COROLLARY: Complex structures on a torus are in (1,1)-correspondence with $GL(2n,\mathbb{Z})\backslash GL(2n,\mathbb{R})/GL(n,\mathbb{C})$.

REMARK: Now I will prove that **the action of $GL(2n,\mathbb{Z})$ on $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$ is ergodic**.

Ergodic group action

DEFINITION: Let (M, μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof. Step 1: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore the set Z_U of such orbits has measure 0.

Proof. Step 2: Choose a countable base $\{U_i\}$ of topology on M. Then the set of points with dense orbits is $M \backslash \bigcup_i Z_{U_i}$. ■

Ergodic complex structures

DEFINITION: Let M be a complex manifold, Teich its Techmüller space, and Γ the mapping group acting on Teich **An ergodic complex structure** is a complex structure with dense Γ-orbit.

CLAIM: Let (M, I) be a manifold with ergodic complex structure, and I' another complex structure. **Then there exists a sequence of diffeomorphisms ν_i such that $\nu_i^*(I)$ converges to I'.**

REMARK: Existence of ergodic complex structures implies that **the moduli space does not exist**. Indeed, the quotient Comp / Diff is a worst topological space ever: **its topology is codiscrete.**

Ergodicity of the mapping class group action

THEOREM: (Calvin C. Moore, 1966) Let Γ be an arithmetic lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact semisimple Lie subgroup. **Then the left action of Γ on G/H is ergodic.** ■

COROLLARY: The action of $GL(2n, \mathbb{Z})$ on $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ is ergodic.

Proof: Indeed, $SL(2n, \mathbb{Z})$ acts on $SL(2n, \mathbb{R})/SL(n, \mathbb{C})$ ergodically by Moore's theorem. ■

THEOREM: Let $M = \mathbb{C}^n/\Lambda$ be a compact torus. **Then M is ergodic if and only if the lattice $\Lambda \cong \mathbb{Z}^{2n}$ is rational.**

Its proof uses Ratner theory.

REMARK: The set of such tori is countable.

Further developments: hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

Ergodicity theorem is true for hyperkähler manifolds: A complex structure on a hyperkähler manifold is ergodic if and only if its Picard rank is maximal.

REMARK: A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

EXAMPLE: Take a 2-dimensional complex torus T, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

DEFINITION: A complex surface is called **a K3 surface** if it a deformation of a Kummer surface. K3 surface is also hyperkähler.

REMARK: Ergodicity theorem is new even for a K3.

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic 2-form.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

REMARK: Usually, one says "hyperkähler manifold" meaning "a compact, Kähler, holomorphically symplectic manifold".

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold **admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.**

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is simple and hyperkähler. Then $C \int_M \eta^{2n} = q(\eta, \eta)^n$, for some primitive integer quadratic form q on $H^2(M, \mathbb{Z})$ and $C > 0$.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by this relation uniquely, up to a sign.**

Computation of the mapping class group

DEFINITION: An arithmetic lattice in a Lie group $G \subset GL(\mathbb{Q}^n)$ is a finite index subgroup in an intersection of G with $GL(\mathbb{Z}^n)$.

Theorem: (Sullivan) Let M be a compact simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M, \mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

COROLLARY: The mapping class group of a compact simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$, is an arithmetic lattice.

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is an arithmetic subgroup** of $O(H^2(M, \mathbb{Z}), q)$.
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M, \mathbb{Z}), q)$ **has finite kernel.**

COROLLARY: Let M be a simple hyperkähler manifold, and Γ its mapping class group. **Then the natural map $\Gamma \longrightarrow O(H^2(M, \mathbb{Z}), q)$ has finite kernel and finite index in $O(H^2(M, \mathbb{Z}), q)$.**

The period map

Remark: For any $J \in$ Teich, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let P : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map P : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: P **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \ | \ \ q(l, l) = 0, \ \ q(l, \bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$

THEOREM: (Bogomolov) Let M be a simple hyperkähler manifold, and Teich its Teichmüller space. **Then the period map P : Teich $\longrightarrow \mathbb{P}er$ is locally a diffeomorphism.**

Global Torelli theorem

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: Let M be a hyperkähler manifold, Teich its Teichmüller space, and Teich_b the quotient of Teich by $\sim$. **Then the period map $P: \text{Teich}_b \longrightarrow \mathbb{P}er$ induces a diffeomorphism on each connected component.**

REMARK: The period space

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l,l) = 0, \quad q(l, \bar{l}) > 0.\}$$

is identified with $Gr_{+,+}(H^2(M, \mathbb{R})) = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$**,** which is a Grassmannian of positive oriented 2-planes in $H^2(M, \mathbb{R})$.

COROLLARY: The mapping class group Γ is an arithmetic subgroup in $G = SO(b_2 - 3, 3)$ acting on $\mathbb{P}er = G/H$, where $H = SO(2) \times SO(b_2 - 3, 1)$. Therefore, **its action is ergodic.**