# Moduli of complex structures and Ratner theory

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## Complex manifolds

**DEFINITION:** Let $M$ be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

**DEFINITION:** An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case $I$ is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

**THEOREM:** (Newlander-Nirenberg)
**This definition is equivalent to the usual one.**

**DEFINITION:** **The space of almost complex structures** is an infinite-dimensional Fréchet manifold $X_M$ of all tensors $I^2 = -\operatorname{Id}_{TM}$, equipped with the natural Fréchet topology.

**CLAIM:** The space Comp of integrable almost complex structures **is a submanifold in $X_M$** (also infinite-dimensional).

## Teichmüller space

**Definition:** Let $M$ be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on $M$, and let $\operatorname{Teich} := \operatorname{Comp} / \operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

**REMARK:** Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

**DEFINITION:** Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of $M$. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **moduli space of complex structures on $M$** is a connected component of $\operatorname{Teich}/\Gamma$.

**REMARK:** This terminology is **standard for curves.**

**REMARK:** The topology of the moduli space $\operatorname{Teich}/\Gamma$ is often bizzarre. However, **its points are in bijective correspondence with equivalence classes of complex structures.**

## Kähler manifolds

**DEFINITION:** A Riemannian metric $g$ on a complex manifold $(M,I)$ is called **Hermitian** if $g(Ix,Iy) = g(x,y)$. In this case, $g(x,Iy) = g(Ix,I^2y) = -g(y,Ix)$, hence $\omega(x,y) := g(x,Iy)$ is skew-symmetric.

**DEFINITION:** The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of $(M,I,g)$.

**DEFINITION:** A complex Hermitian manifold $(M,I,\omega)$ is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form $\omega$ is called **the Kähler class** of $M$, and $\omega$ **the Kähler form**.

**Definition:** Let $M = \mathbb{C}P^n$ be a complex projective space, and $g$ a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

**Remark:** For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. **Fubini-Study form on $T_x\mathbb{C}P^n = \mathbb{C}^n$ is $U(n)$-invariant, hence unique up to a constant.**

## Kähler manifolds II.

**Claim: Fubini-Study form is Kähler.** Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish, because $-\operatorname{Id} \in U(n)$

**REMARK: The same argument works for all symmetric spaces.**

**Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler.** Indeed, a restriction of a closed form is again closed.

**DEFINITION:** The cohomology class of the Kähler form is called **the Kähler class** of a manifold.

**Hodge theory for Kähler manifolds (first cohomology):**
Let $M$ be a compact Kähler manifold, and $\theta \in \Omega^1(M)$ a holomorphic differential. Then $\theta$ is closed, and its cohomology class is non-zero. This gives an injective map $\Psi : H^0(\Omega^1 M) \hookrightarrow H^1(M, \mathbb{C})$. Moreover, **any $\alpha \in H^1(M, \mathbb{C})$ can be decomposed as $\alpha = \alpha^{1,0} + \alpha^{0,1}$, with $\alpha^{1,0} \in \operatorname{im}\Psi$ and $\overline{\alpha^{0,1}} \in \operatorname{im}\Psi$ represented by holomorphic differentials holomorphic.**

**DEFINITION: The space $\operatorname{im}\Psi$ is denoted $H^{1,0}(M)$.**

## Teichmüller space for a compact torus

**DEFINITION:** Let $\mathbb{Z}^{2n} \subset \mathbb{C}^n$ be a cocompact lattice. Then $\mathbb{C}^n/\mathbb{Z}^{2n}$ is a complex manifold, called **a (compact) complex torus**.

**REMARK:** The space of complex structures on $R^{2n}$ is naturally identified with $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$.

**THEOREM:** **Any connected component of the Teichmüller space for a compact torus is identified with $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$.**

**Proof:** Let the **period map** put $(M,I)$ to $H^{1,0}(M) \subset H^1(M,\mathbb{C})$, considered as a point on $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$. **Since $M = H^{1,0}(M)/H^1(M,\mathbb{Z})$, this map is invertible.** ■

**COROLLARY:** **Complex structures on a torus are in (1,1)-correspondence with $GL(2n,\mathbb{Z})\backslash GL(2n,\mathbb{R})/GL(n,\mathbb{C})$.**

**REMARK:** Now I will prove that **the action of $GL(2n,\mathbb{Z})$ on $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$ is ergodic**.

## Ergodic complex structures

**DEFINITION:** Let $(M, \mu)$ be a space with measure, and $G$ a group acting on $M$ preserving measure. This action is **ergodic** if all $G$-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

**CLAIM:** Let $M$ be a manifold, $\mu$ a Lebesgue measure, and $G$ a group acting on $(M, \mu)$ ergodically. **Then the set of non-dense orbits has measure 0.**

**Proof:** Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting $U$, $x \in M \backslash M'$. Therefore the set of such orbits has measure 0. ■

**DEFINITION:** Let $M$ be a complex manifold, Teich its Techmüller space, and Γ the mapping group acting on Teich. **An ergodic complex structure** is a complex structure with dense Γ-orbit.

**CLAIM:** Let $(M, I)$ be a manifold with ergodic complex structure, and $I'$ another complex structure. **Then there exists a sequence of diffeomorphisms $\nu_i$ such that $\nu_i^*(I)$ converges to $I'$.**

## Ergodicity of the mapping class group action

**THEOREM:** (Calvin C. Moore, 1966) Let $\Gamma$ be an arithmetic lattice in a non-compact simple Lie group $G$ with finite center, and $H \subset G$ a non-compact semisimple Lie subgroup. **Then the left action of $\Gamma$ on $G/H$ is ergodic.** ■

**COROLLARY: The action of $GL(2n, \mathbb{Z})$ on $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ is ergodic.**

**Proof:** Indeed, $SL(2n, \mathbb{Z})$ acts on $SL(2n, \mathbb{R})/SL(n, \mathbb{C})$ ergodically by Moore's theorem. ■

**THEOREM:** Let $M = \mathbb{C}^n/\Lambda$ be a compact torus. **Then $M$ is non-ergodic if and only if the lattice $\Lambda \cong \mathbb{Z}^{2n}$ is rational.**

**Its proof uses Ratner theory.**

**REMARK: The set of such tori is countable.**

## Ratner's theorem: preparatory definitions

**DEFINITION:** A matrix is called **unipotent** if it is an exponent of a nilpotent, and **semisimple** if it is conjugate to a diagonal matrix. An element $g$ in an algebraic group is called **unipotent (semisimple)** if it is represented by a unipotent (algebraic) matrix for some algebraic representation.

**THEOREM: (Chevalley-Jordan decomposition)**
Any element $g$ of a Lie algebra **can be represented as $g = s + u$, where $s$ is semisimple, $u$ nilpotent, and $s, u$ commute.** Moreover, **such a decomposition is unique.**

**DEFINITION:** Let $G$ be a connected Lie group equipped with a Haar measure. **A lattice** $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, $G/\Gamma$ has finite volume).

**THEOREM: (Borel and Harish-Chandra)**
An arithmetic subgroup of a reductive algebraic group $G$, defined over $\mathbb{Q}$, **is a lattice whenever $G$ has no non-trivial characters over $\mathbb{Q}$.**

## Ratner's theorem

**THEOREM: (Ratner's theorem)** Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ a lattice. Then **a closure of any $H$-orbit $Hx$ in $G/\Gamma$ is an orbit of a closed, connected subgroup $S \subset G$, such that $S^x \cap \Gamma \subset S$ is a lattice.** Here $S^x = xSx^{-1}$.

**EXAMPLE:** Let $\Lambda \in \mathbb{C}^n$ be a cocompact lattice. The corresponding complex torus is non-ergodic if and only if there exists an intermediate Lie group $H = SL(n, \mathbb{C}) \subset S \subsetneq SL(2n, \mathbb{R})$ such that $S \cap SL(\Lambda)$ is a lattice. **This is equivalent to $S$ being a rational Lie group**, with respect to the rational structure induced by $\Lambda$.

**CLAIM:** Let $n \geqslant 2$, $G = SL(2n, \mathbb{R})$, and $H \cong SL(n, \mathbb{C}) \subset G$. **Then any closed connected Lie subgroup $S \subset G$ containing $H$ coincides with $G$ or with $H$.**

**Proof:** See the next slide. ■

**COROLLARY:** For any non-ergodic torus $\mathbb{C}^n/\Lambda$, the intersection $SL(n, \mathbb{C}) \cap SL(\Lambda)$ is a lattice. **This is equivalent to $\Lambda$ being rational**.

## Intermediate subgroups $SL(n,\mathbb{C}) \subset S \subset SL(2n,\mathbb{R})$

**CLAIM:** Let $n \geqslant 2$. **Then any closed, connected Lie subgroup $S \subset GL(2n,\mathbb{R})$ containing $GL(n,\mathbb{C})$ coincides with $GL(2n,\mathbb{R})$ or with $GL(n,\mathbb{C})$.**

**Proof. Step 0:** $\dim_{\mathbb{R}} GL(n,\mathbb{C}) = 2n^2$, and $\dim_{\mathbb{R}} SL(2n,\mathbb{R}) = 4n^2$.

**Step 1:** It suffices to prove the same result for Lie algebras. Let $\mathfrak{h} = \mathfrak{gl}(n,\mathbb{C}) \subset \mathfrak{s} \subset \mathfrak{gl}(2n,\mathbb{R}) = \mathfrak{g}$. As a representation of $\mathfrak{h}$, the space $\mathfrak{g}$ is a direct sum of $\mathfrak{h}$ (matrices commuting with $I$) and and the space $\mathfrak{g}_-$ of matrices anticommuting with $I$. The later representation is isomorphic to $\mathfrak{h}$; the isomorphism is provided by $a \longrightarrow au$, for any non-degenerate matrix $u$ anticommuting with $I$.

**Step 2:** The subalgebra $\mathfrak{s}$ must be $\mathfrak{h}$-invariant. Therefore, it is either $\mathfrak{h}$ or $\mathfrak{h} \oplus \mathfrak{g}_- = \mathfrak{g}$. ■

## Further developments: hyperkähler manifolds

**Ergodicity theorem is true for hyperkähler manifolds:** A complex structure on a hyperkähler manifold is non-ergodic if and only if its Picard rank is maximal.

**DEFINITION:** A **hyperkähler structure** on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**REMARK: A hyperkähler manifold is holomorphically symplectic:** $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on $(M, I)$.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

**EXAMPLE:** Take a 2-dimensional complex torus $T$, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

**DEFINITION:** A complex surface is called **a K3 surface** if it a deformation of a Kummer surface. K3 surface is also hyperkähler.

## Further developments: Kobayashi non-hyperbolicity

**DEFINITION: An entire curve** is a non-constant map $\mathbb{C} \longrightarrow M$.

**DEFINITION:** A compact complex manifold $M$ is called **Kobayashi hyperbolic** if there exist no entire curves $\mathbb{C} \longrightarrow M$.

Using ergodicity, the following longstanding conjecture was proven.

**THEOREM: All hyperkähler manifolds are non-hyperbolic.**

**REMARK:** This is equivalent to having an entire curve $\mathbb{C} \longrightarrow M$ (Brody).

## Exercises

**EXERCISE:** Let $H \subset G$ be a subgroup. Prove that there are no closed, connected subgroups $S$ satisfying $H \subsetneq S \subsetneq G$ when

a. $H = SO(p,k) \times SO(q-k)$, $G = SO(p,q)$, $p, q > 0$

b. $H = Sp(2n,\mathbb{R})$, $G = SL(2n,\mathbb{R})$

c. $H = SL(n,\mathbb{H})$, $G = SL(2n,\mathbb{C})$

**EXERCISE:** Find all dense $G_{\mathbb{Z}}$-orbits in $G/H$ for all these cases.

**EXERCISE:** Find a K3 surface with vanishing Kobayashi pseudodistance.