Teichmüller space and Ratner theory, lecture 2: Global Torelli theorem for hyperkähler manifolds

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Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I : TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

DEFINITION: **The space of almost complex structures** is an infinite-dimensional Fréchet manifold X_M of all tensors $I^2 = -\operatorname{Id}_{TM}$, equipped with the natural Fréchet topology.

CLAIM: The space Comp of integrable almost complex structures **is a submanifold in X_M** (also infinite-dimensional).

Teichmüller space

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let $\operatorname{Teich} := \operatorname{Comp} / \operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

REMARK: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

DEFINITION: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **moduli space of complex structures on M** is a connected component of $\operatorname{Teich}/\Gamma$.

REMARK: This terminology is **standard for curves.**

REMARK: The topology of the moduli space $\operatorname{Teich}/\Gamma$ is often bizzarre. However, **its points are in bijective correspondence with equivalence classes of complex structures.**

REMARK: To describe the moduli space, we shall compute Teich and Γ.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$**,** $\omega_J := g(J\cdot, \cdot)$**,** $\omega_K := g(K\cdot, \cdot)$**.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: **A hyperkähler manifold can be defined as a manifold which has holonomy in** $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

EXAMPLES.

EXAMPLE: An even-dimensional complex vector space.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: A non-compact example: $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity** $\mathbb{C}^2/\pm 1$**.**

EXAMPLE: Take a 2-dimensional complex torus T, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

REMARK: Take a symmetric square $\operatorname{Sym}^2 T$, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. **Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $\widetilde{T/\pm 1}$.**

DEFINITION: A complex surface is called **K3 surface** if it a deformation of the Kummer surface.

THEOREM: (a special case of Enriques-Kodaira classification)
Let M be a compact complex surface which is hyperkähler. **Then M is either a torus or a K3 surface.**

Hilbert schemes

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Fujiki, Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLE: **A Hilbert scheme of K3** is hyperkähler.

EXAMPLE: Let T be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, a universal covering of $T^{[n]}/T$ is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known compact hyperkaehler manifolds are these 2 and the three series:** tori, Hilbert schemes of K3, and generalized Kummer.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - \frac{n-1}{n} \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n}\right) \left(\int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: q **has signature** $(b_2 - 3, 3)$. It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

Automorphisms of cohomology.

THEOREM: Let M be a simple hyperkähler manifold, and $G \subset GL(H^*(M))$ a group of automorphisms of its cohomology algebra preserving the Pontryagin classes. Then G acts on $H^2(M)$ **preserving the BBF form.** Moreover, the map $G \longrightarrow O(H^2(M,\mathbb{R}), q)$ **is surjective on a connected component, and has compact kernel.**

Proof. Step 1: Fujiki formula $v^{2n} = q(v,v)^n$ implies that Γ_0 **preserves the Bogomolov-Beauville-Fujiki up to a sign.** The sign is fixed, if n is odd.

Step 2: For even n, the sign is also fixed. Indeed, G preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2} \wedge p_1(M) = q(v,v)^{n-1}c$, for some $c \in \mathbb{R}$. The constant c is positive, **because the degree of $c_2(B)$ is positive** for any Yang-Mills bundle with $c_1(B) = 0$.

Step 3: $\mathfrak{o}(H^2(M,\mathbb{R}), q)$ acts on $H^*(M,\mathbb{R})$ by derivations preserving Pontryagin classes (V., 1995). Therefore Lie(G) surjects to $\mathfrak{o}(H^2(M,\mathbb{R}), q)$.

Step 4: The kernel K of the map $G \longrightarrow G\big|_{H^2(M,\mathbb{R})}$ is compact, because it commutes with the Hodge decomposition and Lefschetz $\mathfrak{sl}(2)$-action, hence preserves the Riemann-Hodge form, which is positive definite. ■

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

Proof: Follows from Sullivan and a computation of $\operatorname{Aut}(H^*(M,\mathbb{R}))$ done earlier. ■

REMARK: (Kollar-Matsusaka, Huybrechts) **There are only finitely many connected components** of Teich.

REMARK: The mapping class group acts on the set of connected components of Teich.

COROLLARY: Let Γ_I be the group of elements of mapping class group preserving a connected component of Teichmüller space containing $I \in$ Teich. **Then Γ_I is also arithmetic.** Indeed, **it has finite index in Γ.**

Deformations of holomorphically symplectic manifolds.

THEOREM: (Kodaira) **A small deformation of a compact Kähler manifold is again Kähler.**

COROLLARY: A small deformation of a holomorphically symplectic Kähler manifold M **is again holomorphically symplectic and Kähler.**

Proof: A small deformation M' of M would satisfy $H^{2,0}(M') = H^{2,0}(M)$; however, a small deformation of a non-degenerate (2,0)-form remains non-degenerate. ■

DEFINITION: A compact complex manifold admitting holomorphically symplectic and Kähler structure is called **a manifold of hyperkähler type**

REMARK: By the **Teichmüller space** of hyperkähler manifolds we shall understand the deformation space of complex manifolds of hyperkähler type.

The period map

Definition: Let P : Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map P : Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

REMARK: P **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid \ q(l,l) = 0, q(l,\bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2-3,3)/SO(2) \times SO(b_2-3,1)$

THEOREM: Let M be a simple hyperkähler manifold, and Teich a component of its Teichmüller space. Then
(i) (Bogomolov) **The period map** P: **Teich** $\longrightarrow \mathbb{P}er$ **is etale.**
(ii) (Huybrechts) It is **surjective.**

REMARK: Bogomolov's theorem implies that Teich **is smooth.** It is **non-Hausdorff** even in the simplest examples.

Hausdorff reduction

REMARK: **A non-Hausdorff manifold** is a topological space locally diffeomorphic to $\mathbb{R}^n$.

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (D. Huybrechts) If I_1, $I_2 \in$ Teich are non-separable points, then $P(I_1) = P(I_2)$, and (M, I_1) **is birationally equivalent** to (M, I_2)

DEFINITION: Let M be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called **a Hausdorff reduction** of M.

Problems:
Generally speaking,
1. **$\sim$ is not always an equivalence relation.**
2. Even if $\sim$ is equivalence, the $M/\sim$ is not always Hausdorff.

REMARK: Huybrechts's theorem implies that $\sim$ **is in fact an equivalence relation;** the quotient is mapped to the period space by etale map, hence **it is Hausdorff.**

Global Torelli theorem

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ **is an isomorphism,** for each connected component of Teich_b.

The proof is based on metric geometry and twistor construction.

Sub-Riemannian structures

DEFINITION: Let M be a Riemannian manifold and $B \subset TM$ a sub-bundle. A **horizontal path** is a piecewise smooth path $\gamma : [b,a] \longrightarrow M$ tangent to B everywhere. A **sub-Riemannian**, or **Carno-Carathéodory** metric M is

$$d_B(x,y) := \inf_{\gamma \text{ horizontal}} L(\gamma) :$$

the infimum of the length $L(\gamma)$ for all horizontal paths connecting x to y.

THEOREM: (Chow-Rashevskii theorem; 1938, 1939)
Consider **the Frobenius form** $\Phi : \Lambda^2 B \longrightarrow TM/B$ mapping vector fields $X, Y \in B$ to an image of $[X,Y]$ modulo B. Suppose that Φ is surjective. **Then any two points can be connected by a horizontal path,** and the sub-Riemannian metric d_B is finite.

Properties of sub-Riemannian metrics

Let (M, B, g) be a sub-Riemannian manifold.

CLAIM: Every two points $x, y \in M$ are connected by a smooth, horizontal path γ. Moreover, $d_B(x, y) = \inf_{\gamma \text{ horizontal, smooth}} L(\gamma)$: the sub-Riemannian distance can be taken as infimum of the length for smooth horizontal paths connecting x to y.

THEOREM: (ball-box theorem) An ε-ball in d_B is **asymptotically equivalent to a product of ε-ball in direction of B and ε^2-ball in orthogonal direction.**

COROLLARY: The sub-Riemannian metric **induces the standard topology on M.**

COROLLARY: The Hausdorff dimension of a sub-Riemannian manifold is integer, **and strictly bigger than** $\dim M$**.**

Subtwistor metric

Let H be a real vector space with non-degenerate scalar product of signature $(3, b-3)$, and $\mathrm{Gr}_{++}(H)$ the Grassmannian of 2-dimensional positive oriented planes in H. The space $\mathrm{Gr}_{++}(H)$ is in fact a complex manifold, and it is called **the period space of weight 2 Hodge structures on H.**

DEFINITION: Let $W \subset V$ be a positive 3-dimensional subspace, and $S_W = \mathrm{Gr}_{++}(W) \subset \mathrm{Gr}_{++}(H)$ a 2-dimensional sphere consisting all 2-dimensional oriented planes in W. Then S_w is called **a twistor line**.

CLAIM: Each pair $x, y \in \mathrm{Gr}_{++}(H)$ can be connected by an intersecting chain $S_{W_1}, S_{W_2}, ..., S_{W_n}$ of twistor lines; moreover, $n \leqslant 3$.

DEFINITION: A twistor path on $\mathrm{Gr}_{++}(H)$ is a piecewise smooth path $\gamma: [a, b] \longrightarrow \mathrm{Gr}_{++}(H)$ with each smooth component sitting on a twistor line.

DEFINITION: Fix a Euclidean structure on H, and let g be the corresponding Riemannian metric on $\mathrm{Gr}_{++}(H)$. **Subtwistor metric** $d_{tw}(x, y)$ on $\mathrm{Gr}_{++}(H)$ is defined as $d_{tw}(x, y) := \inf_\gamma L(\gamma)$ where $L(\gamma)$ is a length of the path γ taken with respect to g, and infimum is taken over all subtwistor paths connecting x to y.

Properties of subtwistor metric

QUESTION: Can we connect any pair $x, y \in \mathrm{Gr}_{++}(H)$ with a smooth path tangent to twistor line at each point? Would the infimum of its length give the same metric?

QUESTION: What about the ball-box theorem? What is a shape of a small ε-ball in d_{tw}?

QUESTION: What us the Hausdorff dimension $(\mathrm{Gr}_{++}(H), d_{tw})$?

Theorem 1: The subtwistor metric d_{tw} induces the standard topology on $\mathrm{Gr}_{++}(H)$.

REMARK: Global Torelli theorem immediately follows from this (non-trivial) statement.

Twistor lines

REMARK: The period space

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid \ q(l,l) = 0, \quad q(l,\bar{l}) > 0.\}$$

is identified with $Gr_{++}(H^2(M,\mathbb{R})) = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$**,** which is a Grassmannian of positive oriented 2-planes in $H^2(M,\mathbb{R})$.

DEFINITION: Given a hyperkähler structure (M,I,J,K) on a hyperkähler manifold, and $a,b,c \in \mathbb{R}, a^2+b^2+c^2=1$, the operator $L := aI+bJ+cK$ defines a complex structure on M. We call L **an induced compex structure**. The set of all induced complex structures defines a deformation of the complex structure on M, called **the twistor deformation**, and its total space is **the twistor space**.

REMARK: Image of a twistor deformation in $\mathbb{P}$**er is a twistor line.**

Kähler cone and Hodge decomposition

REMARK: Let (M,I) be a hyperkähler manifold with $\operatorname{Per}(I) = V \in \operatorname{Gr}_{++}(H^2(M,\mathbb{R}))$. Then $H^{1,1}(M,I) = V^\perp$.

CLAIM: (Huybrechts, Boucksom)
Let (M,I) be a hyperkaehler manifold with $\operatorname{NS}(M,I) := H^{1,1}_I(M,\mathbb{Z}) = 0$. **Then the Kahler cone of (M,I) is one of two components of the positive cone in $H^{1,1}_I(M)$.**

DEFINITION: Let $S \subset \operatorname{Teich}$ be a $\mathbb{C}P^1$ associated with a twistor family. It is called **generic** if it passes through a point $I \in \operatorname{Teich}$ with $\operatorname{NS}(M,I) = 0$.

DEFINITION: A hyperkähler 3-plane $W \subset H^2(M,\mathbb{R})$ is called **generic** if $W^\perp \cap H^2(M,\mathbb{Z}) = 0$. The corresponding $\mathbb{C}P^1 \subset \mathbb{P}\mathrm{er}$ in the period space is called **a GHK line**.

REMARK: These two notions are equivalent.

Lifting property for GHK lines

REMARK: Consider a 3-plane $W = \langle \omega_I, \omega_J, \omega_K \rangle$ associated with a hyperkähler structure, and let S be the set of oriented 2-planes in W. Denote by S_{ng} the set of $x \in S$ satisfying $x^\perp \cap H^2(M, \mathbb{Z}) \neq 0$. If W is generic, then S_{ng} is countable.

THEOREM: (A lifting property for GHK lines)
Let $W \subset H^2(M, \mathbb{R})$ be a generic 3-plane, and $S \subset \mathbb{P}\mathrm{er}$ the corresponding GHK line. Consider the period map $P: \mathrm{Teich} \longrightarrow \mathbb{P}\mathrm{er}$. **Then $P^{-1}(S)$ is a union of a countable set mapped to S_{ng}, and a disconnected set of rational curves bijectively mapped to S.**

Proof. Step 1: Let $x \notin S_{ng}$ We are going to prove that for all $I \in P^{-1}(x)$, y **is contained in a connected component of $P^{-1}(S)$, bijectively mapped to S.**

Step 2: Notice that $NS(I) = x^\perp \cap H^2(M, \mathbb{Z}) = 0$. Therefore the Kähler cone of (M, I) **is one of two components of the set $\{\omega \in P(I)^\perp \ \mid \ q(\omega, \omega) > 0\}$.**.

Step 3: For each positive 3-plane $W \subset H^2(M,\mathbb{R})$, $W = \langle \omega_I, \omega_J, \omega_K \rangle$ for some hyperkähler structure I, J, K. **Then the twistor family associated with I, J, K is mapped to S.** ■

COROLLARY: Any connected sequence of twistor lines in $\mathbb{P}$er intersecting in points I_k with $NS(I_k) = 0$ can be lifted to a connected sequence in $\mathbb{P}$er. We call twistor paths on these sequences **GHK twistor paths.**

Subtwistor metric on the Teichmüller space

DEFINITION: Subtwistor metric on Teich is a metric induced from $\mathbb{P}\mathrm{er}$.

CLAIM: Per : Teich $\longrightarrow$ $\mathbb{P}\mathrm{er}$ **is a surjective isometry.**

Proof: For any irrational 3-dimensional space $W \in \mathrm{Gr}_{+++}(H^2(M,\mathbb{R}))$, the corresponding twistor line S_W is lifted wholly to $\mathbb{P}\mathrm{er}$; for each irrational point $I \in S_W$, this lifting is determined by a preimage $I_1 \in \mathrm{Per}^{-1}(I)$ uniquely. Any twistor path can be approximated by one which is obtained from irrational $W \in \mathrm{Gr}_{+++}(H^2(M,\mathbb{R}))$, hence the map Per : Teich $\longrightarrow$ $\mathbb{P}\mathrm{er}$ is an isometry. ■

Global Torelli theorem follows immediately. Indeed, by Theorem 1, any twistor paths in $\mathbb{P}\mathrm{er}$ can be approximated by GHK twistor paths in usual, and hence in subtwistor, topology. The latter are liftable to Teich. This implies that **the subtwistor distance in** Teich **is equal to that in** $\mathbb{P}\mathrm{er}$**.**

Exercises

EXERCISE: Let Symp be the set of all symplectic structures on a symplectic manifold, and $\operatorname{Teich}_s = \operatorname{Symp}/\operatorname{Diff}_0$. Consider the period map $\operatorname{Per}_s: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$. **Prove that Per_s is locally a diffeomorphism.**

EXERCISE: Let Hyp be the set of all hyperkäher structures (I,J,K,g) on a symplectic manifold, and $\operatorname{Teich}_h = \operatorname{Hyp}/\operatorname{Diff}_0$. Consider the period map $\operatorname{Per}_h: \operatorname{Teich}_h \longrightarrow H^2(M,\mathbb{R})^3$ mapping (I,J,K,g) to the cohomology classes of $\omega_I, \omega_J, \omega_K$. **Prove that Per_h is locally a diffeomorphism from Teich_h to the set of pairwise orthogonal triples $x,y,z \in H^2(M,\mathbb{R})$ satisfying $q(x,x) = q(y,y) = q(z,z) > 0$.**

EXERCISE: Show that $\operatorname{Teich}_h/\Gamma$ is Hausdorff.