Ratner’s theorem
and ergodic complex structures
on hyperkaehler manifolds

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Teichmüller spaces

**Definition:** Let $M$ be a compact complex manifold, and $\text{Diff}_0(M)$ a connected component of its diffeomorphism group (the group of isotopies). Denote by $\text{Comp}$ the space of complex structures on $M$, and let $\text{Teich} := \text{Comp} / \text{Diff}_0(M)$. We call it the **Teichmüller space**.

**Remark:** $\text{Teich}$ is a finite-dimensional complex space (Kodaira-Spencer-Kuranishi-Douady), but often non-Hausdorff.

**Definition:** Let $\text{Diff}_+(M)$ be the group of oriented diffeomorphisms of $M$. We call $\Gamma := \text{Diff}_+(M) / \text{Diff}_0(M)$ the **mapping class group**. The coarse moduli space of complex structures on $M$ is a connected component of $\text{Teich} / \Gamma$.

**Remark:** This terminology is standard for curves.
Moduli spaces

**DEFINITION:** The quotient $\text{Comp}/\text{Diff} = \text{Teich}/\Gamma$ is called the moduli space of complex structures. Typically, it is very non-Hausdorff. $\text{Comp}/\text{Diff}$ corresponds bijectively to the set of isomorphism classes of complex structures.

**REMARK:** The moduli space exists, and is quasiprojective, for curves and manifolds with canonical polarization (Viehweg, Schumacher). The moduli space exists as a non-Hausdorff algebraic space when $M$ is Kähler and $H^2(M) = H^{1,1}(M)$: Calabi-Yau manifolds, generalized Enriques manifolds, rational manifolds (Viehweg).

This talk is about an opposite situation, when $\Gamma$ acts on $\text{Teich}$ ergodically.
Holomorphically symplectic manifolds

**DEFINITION:** A hyperkähler structure on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**REMARK:** A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1} \omega_K$ is a holomorphic symplectic form on $(M, I)$.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**DEFINITION:** For the rest of this talk, a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.

**DEFINITION:** A compact hyperkähler manifold $M$ is called simple if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

**Bogomolov’s decomposition:** Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

**Further on,** all hyperkähler manifolds are assumed to be simple.
The Bogomolov-Beauville-Fujiki form

**THEOREM:** (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where $M$ is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta, \eta)^n$, for some primitive integer quadratic form $q$ on $H^2(M, \mathbb{Z})$, and $c > 0$ an integer number.

**Definition:** This form is called **Bogomolov-Beauville-Fujiki form**. It is defined by the Fujiki’s relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville)

$$
\lambda q(\eta, \eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - 
- \frac{n-1}{n} \left( \int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n} \right) \left( \int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1} \right)
$$

where $\Omega$ is the holomorphic symplectic form, and $\lambda > 0$. 
Computation of the mapping class group

**Theorem:** (Sullivan) Let $M$ be a compact, simply connected Kähler manifold, $\dim \mathbb{C} M \geq 3$. Denote by $\Gamma_0$ the group of automorphisms of an algebra $H^*(M, \mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then the natural map $\text{Diff}_+(M)/\text{Diff}_0 \to \Gamma_0$ has finite kernel, and its image has finite index in $\Gamma_0$.

**Theorem:** Let $M$ be a simple hyperkähler manifold, and $\Gamma_0$ as above. Then
(i) $\Gamma_0|_{H^2(M, \mathbb{Z})}$ is a finite index subgroup of $O(H^2(M, \mathbb{Z}), q)$.
(ii) The map $\Gamma_0 \to O(H^2(M, \mathbb{Z}), q)$ has finite kernel.
The period map

Remark: For any $J \in \text{Teich}$, $(M, J)$ is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $P : \text{Teich} \rightarrow \mathbb{P}H^2(M, \mathbb{C})$ be a map $J$ to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $P : \text{Teich} \rightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called the period map.

Remark: $P$ maps Teich into an open subset of a quadric, defined by

$$\text{Per} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}.$$ 

It is called the period space of $M$. 


Period space as a Grassmannian of positive 2-planes

PROPOSITION: The period space

\[ \text{Per} := \{ l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0 \} \]

is identified with \( SO(b_2-3, 3)/SO(2) \times SO(b_2-3, 1) \), which is a Grassmannian of positive oriented 2-planes in \( H^2(M, \mathbb{R}) \).

Proof. Step 1: Given \( l \in \mathbb{P}H^2(M, \mathbb{C}) \), the space generated by \( \text{Im} l, \text{Re} l \) is 2-dimensional, because \( q(l, l) = 0, q(l, \bar{l}) \) implies that \( l \cap H^2(M, \mathbb{R}) = 0 \).

Step 2: This 2-dimensional plane is positive, because \( q(\text{Re} l, \text{Re} l) = q(l + \bar{l}, l + \bar{l}) = 2q(l, \bar{l}) > 0 \).

Step 3: Conversely, for any 2-dimensional positive plane \( V \in H^2(M, \mathbb{R}) \), the quadric \( \{ l \in V \otimes_{\mathbb{R}} \mathbb{C} \mid q(l, l) = 0 \} \) consists of two lines; a choice of a line is determined by orientation. ■
Birational Teichmüller moduli space

**DEFINITION:** Let $M$ be a topological space. We say that $x, y \in M$ are non-separable (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

**THEOREM:** (Huybrechts) Two points $I, I' \in \text{Teich}$ are non-separable if and only if there exists a bimeromorphism $(M, I) \to (M, I')$ which is non-singular in codimension 2.

**DEFINITION:** The space $\text{Teich}_b := \text{Teich} / \sim$ is called the birational Teichmüller space of $M$.

**THEOREM:** The period map $\text{Teich}_b \xrightarrow{\text{Per}} \mathbb{P} \text{er}$ is an isomorphism, for each connected component of $\text{Teich}_b$.

**DEFINITION:** Let $M$ be a hyperkaehler manifold, $\text{Teich}_b$ its birational Teichmüller space, and $\Gamma$ the mapping class group. The quotient $\text{Teich}_b / \Gamma$ is called the birational moduli space of $M$. 
Monodromy group and the birational moduli space

THEOREM: Let \((M, I)\) be a hyperkähler manifold, and \(W\) a connected component of its birational moduli space. Then \(W\) is isomorphic to \(\mathbb{P}_{\text{Per}}/\Gamma\), where \(\mathbb{P}_{\text{Per}} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)\) and \(\Gamma\) is an arithmetic group in \(O(H^2(M, \mathbb{R}), q)\), called the monodromy group.

REMARK: \(\Gamma_I\) is a group generated by monodromy of the Gauss-Manin local system on \(H^2(M)\).

A CAUTION: Usually “the global Torelli theorem” is understood as a theorem about Hodge structures. For K3 surfaces, the Hodge structure on \(H^2(M, \mathbb{Z})\) determines the complex structure. For \(\dim_{\mathbb{C}} M > 2\), it is false.

REMARK: Further on, I shall freely identify \(\mathbb{P}_{\text{Per}}\) and \(\text{Teich}_b\).
Ergodic complex structures

**DEFINITION:** Let \((M, \mu)\) be a space with measure, and \(G\) a group acting on \(M\) preserving measure. This action is **ergodic** if all \(G\)-invariant measurable subsets \(M' \subset M\) satisfy \(\mu(M') = 0\) or \(\mu(M \setminus M') = 0\).

**CLAIM:** Let \(M\) be a manifold, \(\mu\) a Lebesgue measure, and \(G\) a group acting on \((M, \mu)\) ergodically. Then the set of non-dense orbits has measure 0.

**Proof:** Consider a non-empty open subset \(U \subset M\). Then \(\mu(U) > 0\), hence \(M' := G \cdot U\) satisfies \(\mu(M \setminus M') = 0\). For any orbit \(G \cdot x\) not intersecting \(U\), \(x \in M \setminus M'\). Therefore the set of such orbits has measure 0. □

**DEFINITION:** Let \(M\) be a complex manifold, Teich its Teichmüller space, and \(\Gamma\) the mapping group acting on Teich. An ergodic complex structure is a complex structure with dense \(\Gamma\)-orbit.

**CLAIM:** Let \((M, I)\) be a manifold with ergodic complex structure, and \(I'\) another complex structure. Then there exists a sequence of diffeomorphisms \(\nu_i\) such that \(\nu_i^*(I)\) converges to \(I'\).
Ergodic complex structures

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Ergodicity of the monodromy group action

The moduli space \( \mathbb{P}e r / \Gamma_I \) is extremely non-Hausdorff.

**THEOREM:** (Calvin C. Moore, 1966) Let \( \Gamma \) be an arithmetic lattice in a non-compact simple Lie group \( G \) with finite center, and \( H \subset G \) a non-compact subgroup. Then the left action of \( \Gamma \) on \( G/H \) is ergodic.

**THEOREM:** Let \( \mathbb{P}e r \) be a component of a birational Teichmüller space, and \( \Gamma \) its monodromy group. Let \( \mathbb{P}e r_e \) be a set of all points \( L \subset \mathbb{P}e r \) such that the orbit \( \Gamma \cdot L \) is dense (such points are called **ergodic**). Then \( Z := \mathbb{P}e r \setminus \mathbb{P}e r_e \) has measure 0.

**Proof. Step 1:** Let \( G = SO(b_2 - 3, 3), \) \( H = SO(2) \times SO(b_2 - 3, 1) \). Then \( \Gamma \)-action on \( G/H \) is ergodic, by Moore's theorem.

**Step 2:** Ergodic orbits are dense, non-ergodic orbits have measure 0. ■

**REMARK:** This implies that “almost all” \( \Gamma \)-orbits in \( G/H \) are dense.

**REMARK:** Generic deformation of \( M \) has no rational curves, and no non-trivial birational models. Therefore, outside of a measure zero subset, \( \text{Teich} = \text{Teich}_b \). This implies that **almost all complex structures on** \( M \) **are ergodic.**
Ratner's theorem

**DEFINITION:** Let $G$ be a connected Lie group equipped with a Haar measure. A lattice $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, $G/\Gamma$ has finite volume).

**THEOREM:** Let $H \subset G$ be a Lie subgroup generated by unipotents, and $\Gamma \subset G$ a lattice. Then a closure of any $H$-orbit in $G/\Gamma$ is an orbit of a closed, connected subgroup $S \subset G$, such that $S \cap \Gamma \subset S$ is a lattice.

**EXAMPLE:** Let $V$ be a real vector space with a non-degenerate bilinear symmetric form of signature $(3,k)$, $k > 0$ $G := SO^+(V)$ a connected component of the isometry group, $H \subset G$ a subgroup fixing a given positive 2-dimensional plane, $H \cong SO^+(1,k) \times SO(2)$, and $\Gamma \subset G$ an arithmetic lattice. Consider the quotient $\Per := H \backslash G$. Then

A). A point $J \in \Per$ has dense $\Gamma$-orbit if and only if the orbit $H \cdot J$ in the quotient $G/\Gamma$ is closed.

B). A closure of $H \cdot J$ in $G/\Gamma$ is an orbit of a closed connected Lie group $S \supset H$:

$$H \cdot J = S \cdot J \subset \Per.$$
Characterization of ergodic complex structures

CLAIM: Let $G = SO^+(3, k)$, and $H \cong SO^+(1, k) \times SO(2) \subset G$. Then any closed connected Lie subgroup $S \subset G$ containing $H$ coincides with $G$ or with $H$.

COROLLARY: Let $J \in \text{Per} = G/H$. Then either $J$ is ergodic, or its $\Gamma$-orbit is closed in $\text{Per}$.

REMARK: By Ratner's theorem, in the latter case the $H$-orbit of $J$ has finite volume in $G/\Gamma$. Therefore, its intersection with $\Gamma$ is a lattice in $H$. This brings

COROLLARY: Let $J \in \text{Per}$ be a point such that its $\Gamma$-orbit is closed in $\text{Per}$. Consider its stabilizer $\text{St}(J) \cong H \subset G$. Then $\text{St}(J) \cap \Gamma$ is a lattice in $\text{St}(J)$.

COROLLARY: Let $J$ be a non-ergodic complex structure on a hyperkähler manifold, and $W \subset H^2(M, \mathbb{R})$ be a plane generated by $\text{Re} \Omega, \text{Im} \Omega$. Then $W$ is rational. Equivalently, this means that $\text{Pic}(M)$ has maximal possible dimension.

REMARK: This can be used to show that any hyperkähler manifold is Kobayashi non-hyperbolic.
Kobayashi hyperbolic manifolds

**DEFINITION:** An entire curve is a non-constant map $\mathbb{C} \rightarrow M$.

**DEFINITION:** A compact complex manifold $M$ is called Kobayashi hyperbolic, if there exist no entire curves $\mathbb{C} \rightarrow M$.

**THEOREM:** (Brody, 1975)
Let $I_i$ be a sequence of complex structures on $M$ which are not hyperbolic, and $I$ its limit. Then $(M, I)$ is also not hyperbolic.

**THEOREM:** All hyperkähler manifolds are non-hyperbolic.

**REMARK:** This conjecture would follow if we produce an ergodic complex structure which is non-hyperbolic. Indeed, a closure of its orbit is the whole Teich, and a limit of non-hyperbolic complex structures is non-hyperbolic.

**REMARK:** For all known examples of hyperkähler manifolds this result is already proven, due to L. Kamenova and M. V.
**Twistor spaces and hyperkähler geometry**

**DEFINITION:** A **hyperkähler structure** on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**DEFINITION:** **Induced complex structures** on a hyperkähler manifold are complex structures of form $S^2 \cong \{ L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1. \}$

**DEFINITION:** A **twistor space** $\text{Tw}(M)$ of a hyperkähler manifold is a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$. More formally:

Let $\text{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_mM \to T_mM$ on $M$ induced by $J \in S^2 \subset \mathbb{H}$. Let $I_J$ denote the complex structure on $S^2 \cong \mathbb{C}P^1$.

The operator $I_{\text{Tw}} = I_m \oplus I_J : T_x \text{Tw}(M) \to T_x \text{Tw}(M)$ satisfies $I_{\text{Tw}}^2 = -\text{Id}$. **It defines an almost complex structure on $\text{Tw}(M)$**. This almost complex structure is known to be integrable (Obata).
Entire curves in twistor fibers

**THEOREM: (F. Campana, 1992)**
Let $M$ be a hyperkähler manifold, and $\text{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ its twistor projection. Then there exists an entire curve in some fiber of $\pi$.

**CLAIM:** There exists a twistor family which has only ergodic fibers.

**Proof:** There are only countably many complex structures which are not ergodic. ■

**THEOREM:** All hyperkähler manifolds are non-hyperbolic.

**Proof:** Let $\text{Tw}(M) \rightarrow \mathbb{C}P^1$ be a twistor family with all fibers ergodic. By Campana’s theorem, one of these fibers, denoted $(M, I)$, is non-hyperbolic. Since any complex structure $I' \in \text{Teich}$ lies in the closure of $\text{Diff}(M) \cdot I$, all complex structures $I' \in \text{Teich}$ are non-hyperbolic. ■