Ergodic complex structures

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Plan of the talk

1. Introduce Teichmüller spaces and mapping class group. Consider the case of elliptic curve and curve of genus > 1.

2. Compute the Teichmüller space and mapping class group for a complex torus of dimension > 1.

3. Define ergodic actions. Show that the action of the mapping class group of a complex torus of dimension > 1 on its Teichmüller space is ergodic. Relate ergodic actions and dense orbits. Show the density of orbits for complex torus.

4. Introduce hyperkähler manifolds and their moduli. Define **the birational moduli space as a quotient of a Teichmüller space** $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ **by an arithmetic group** Γ_I**.**

2. Explore the non-Hausdorff properties of the birational moduli. Explain how the Moore's ergodic theorem is relevant. Construct an ergodic complex structure on a K3 surface explicitly.

Teichmüller spaces

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let $\operatorname{Teich} := \operatorname{Comp}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on M** is a connected component of $\operatorname{Teich}/\Gamma$.

REMARK: This terminology is **standard for curves.**

Moduli spaces

DEFINITION: The quotient $\operatorname{Comp}/\operatorname{Diff} = \operatorname{Teich}/\Gamma$ is called **the moduli space** of complex structures. Typically, **it is very non-Hausdorff**. Comp corresponds bijectively to the set of isomorphism classes of complex structures.

REMARK: The moduli space exists, and is quasiprojective, for curves and manifolds with canonical polarization (Calabi-Yau, Viehweg, Schumacher). The moduli space exists as a non-Hausdorff algebraic space when M is Kähler and $H^2(M) = H^{1,1}(M)$: Calabi-Yau manifolds, generalized Enriques manifolds, rational manifolds (Viehweg).

Kähler manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

DEFINITION: An Riemannian metric g on a complex manifold (M, I) is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Examples of Kähler manifolds.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n = \mathbb{C}^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: **Fubini-Study form is Kähler.** Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish, because $-\mathrm{Id} \in U(n)$

REMARK: **The same argument works for all symmetric spaces.**

Corollary: **Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler.** Indeed, a restriction of a closed form is again closed.

Sullivan's theorem on the mapping class group

Theorem: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M, \mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map** $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ **has finite kernel, and its image has finite index in** Γ_0**.**

DEFINITION: An algebraic group over a field k is an algebraic variety G of finite type over k equipped with an algebraic map $G \times G \longrightarrow G$ satisfying group axioms. **A Lie group over integers** is the group of integer points in an algebraic group over $\mathbb{Q}$.

DEFINITION: Two groups G, G' are called **commensurable** if G projects with finite kernel to a subgroup of finite index in G'. An **arithmetic group** is a group which is commensurable to a Lie group over integers.

COROLLARY: Let Γ be a mapping class group of a Kähler manifold M, $\dim_{\mathbb{C}} M \geqslant 3$. **Then Γ is an arithmetic group.**

REMARK: Mapping class group of a curve of genus $g > 1$ is not an arithmetic group. Moreover, none of its normal subgroups is arithmetic (Nikolai Ivanov).

Conformal structures in complex dimension 1

DEFINITION: Let $\mathcal{C}^+M$ be the group of positive smooth functions on M, and $R(M)$ the set of all Riemannian metrics. Clearly, $\mathcal{C}^+M$ acts on $R(M)$ by multilication **A conformal structure** on M is a class $c \in R(M)/\mathcal{C}^+M$.

THEOREM: Let X be a real manifold of dimension 2. **There is a bijection between complex structures on X and conformal structures.**

THEOREM: (Riemann) For any conformal structure on X, $\dim_{\mathbb{R}} X = 2$, **there exists a unique, up to a constant multiplier, metric of constant Gaussian curvature in the same conformal class.**

Teichmüller space for $\dim_c = 1$

COROLLARY: The Teichmüller space for X is a space of metrics of constant Gaussian curvature and volume 1, up to isotopies.

COROLLARY: For $g(X) > 1$**,** Teich **is identified with the space of homomorphisms** $\pi_1(X) \xrightarrow{\varphi} \operatorname{Iso}(\Delta)$**,** where Δ is a Poincare disc, $\operatorname{im}\varphi$ acts on Δ properly, and the quotient has volume 1.

COROLLARY: For $g(X) = 1$**,** Teich $= GL(1,\mathbb{C})\backslash GL(2,\mathbb{R})/GL(2,\mathbb{Z})$**.**

I will continue with explanation of this construction for a complex torus of any dimension. This is done using the Calabi-Yau theorem.

Ricci form and curvature

REMARK: When speaking of a **"curvature of a holomorphic bundle",** one usually means the curvature of its Chern connection.

DEFINITION: Let M be a Kähler manifold, $\dim_{\mathbb{C}} M = n$, and $K(M) := \Omega^n M$ its canonical bundle (bundle of holomorphic volume forms). Consider the Chern connection on $K(M)$, and let $\Theta_K \in \Lambda^2(M) \otimes \operatorname{End}(K(M)) = \Lambda^2(M)$ be its curvature. The form Θ_K is called **the Ricci form of M**, and its cohomology class **the first Chern class** of M.

DEFINITION: A manifold is called **Ricci-flat** if its Ricci curvature vanishes.

Calabi-Yau theorem

DEFINITION: Let (M, ω) be a Kähler manifold, ω its Kähler form. Cohomology class of ω is called **the Kähler class** of (M, ω).

THEOREM: (Calabi-Yau)
Let (M, I, g) be Calabi-Yau manifold. **Then there exists a Ricci-flat Kaehler metric on M in any given Kaehler class.** Moreover, such a metric is unique.

THEOREM: ("Lübke vanishing")
Let g be a Ricci-flat metric on a compact complex torus. **Then g is flat,** that is, the Levi-Civita connection of g is flat.

Of course, g depends on the choice of Kähler class in $H^2(M, \mathbb{R})$.

Flat affine structures

DEFINITION: A manifold with flat torsion-free connection ∇ is called **a flat affine manifold**. If ∇ preserves a complex structure, it is called **a flat affine complex manifold**.

THEOREM: A connected, compact **flat affine manifold is a quotient of $U \subset \mathbb{R}^n$ by a group of affine automorphisms acting on U properly.**

COROLLARY: Any flat affine torus is a quotient of $\mathbb{R}^n$ by $\mathbb{Z}^n$ acting by parallel transport.

EXAMPLE: Let $T = \mathbb{C}^n/\mathbb{Z}^{2n}$ be a flat complex torus, and ω a Kähler form. Taking average of ω with respect to the action T, we obtain a flat Kähler metric in the same cohomology class. **It is obviously flat, hence Ricci-flat.**

CLAIM: Let (T, I) be a compact complex torus, and g_1, g_2 – Ricci-flat metrics associated with two different Kähler classes. **Then the corresponding Levi-Civita connections are equal.**

Teichmüller space for a torus

Proof. Step 1: Chose any flat affine connection ∇ on (T,I), compatible with the complex structure, and take the average of g_1 and g_2 with respect to T acting on itself. **We obtain two Ricci-flat metrics g_1' and g_2' in the same Kähler classes, preserved by ∇.**

Step 2: Then ∇ is the Levi-Civita connection for g_1' and g_2' by definition of Levi-Civita connection.

Step 3: Finally, $g_1 = g_1', g_2 = g_2'$, because **Ricci-flat metric is unique in its Kähler class.** ■

COROLLARY: Let (T,I) be a torus. **Then there exists a diffeomorphism from (T,I) to a flat torus.**

REMARK: Using the Ricci flow, **these diffeomorphisms can be chosen continuously on** Comp**.**

COROLLARY: The Teichmüller space of complex structures on $T = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is identified with the space of flat complex structures:

$$\text{Teich}(T) = GL(2n,\mathbb{R})/GL(n,\mathbb{C})$$

Ergodic complex structures

DEFINITION: Let (M, μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore the set of such orbits has measure 0. ■

DEFINITION: Let M be a complex manifold, Teich its Techmüller space, and Γ the mapping group acting on Teich **An ergodic complex structure** is a complex structure with dense Γ-orbit.

CLAIM: Let (M, I) be a manifold with ergodic complex structure, and I' another complex structure. **Then there exists a sequence of diffeomorphisms ν_i such that $\nu_i^*(I)$ converges to I'.**

Ergodic action on homogeneous spaces

THEOREM: (Calvin C. Moore, 1966) Let Γ be an arithmetic lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact subgroup. Then the left action of Γ on G/H is **ergodic,** that is, **for all Γ-invariant measurable subsets $Z \subset G/H$, either Z has measure 0, or $G/H \backslash Z$ has measure 0.**

EXAMPLE: $GL(2n, \mathbb{Z})$-action on $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ is ergodic.

COROLLARY: Let T be a complex compact torus of dimension n, $n > 1$, Teich $= GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ its Teichmüller space, and $\Gamma = GL(2n, \mathbb{Z})$ the mapping class group. **Then any two Γ-invariant orbits intersect.**

QUESTION: Existence of dense $GL(2n, \mathbb{Z})$-orbits in $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ is established. **How to produce an explicit dense orbit?**

CLAIM: (Dmitry Kleinbock) Not all orbits of $GL(4, \mathbb{Z})$-orbits in $GL(4, \mathbb{R})/GL(2, \mathbb{C})$ are dense.

Holomorphically symplectic manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I\circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A compact hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

Hilbert schemes

THEOREM: (a special case of Enriques-Kodaira classification) Let M be a compact complex surface which is hyperkähler. **Then M is either a torus or a K3 surface.**

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLES.

EXAMPLE: A Hilbert scheme of K3 is simple and hyperkähler.

EXAMPLE: Let T be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, a universal covering of $T^{[n]}/T$ is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known simple hyperkaehler manifolds are these 2 and two series:** Hilbert schemes of K3, and generalized Kummer.

REMARK: For hyperkähler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\begin{aligned}\lambda q(\eta,\eta) = & \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ & -\frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)\end{aligned}$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb C} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb Z)$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb Z)}$ **is a finite index subgroup of $O(H^2(M,\mathbb Z),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb Z),q)$ **has finite kernel.**

The period map

Remark: For any $J \in$ Teich, (M,J) is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.

Definition: Let $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

REMARK: P **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l,\bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts) Two points $I, I' \in \text{Teich}$ **are non-separable if and only if there exists a bimeromorphism** $(M, I) \longrightarrow (M, I')$ **which is non-singular in codimension 2.**

DEFINITION: The space $\text{Teich}_b := \text{Teich} / \sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\text{Teich}_b \xrightarrow{\text{Per}} \mathbb{P}\text{er}$ **is an isomorphism,** for each connected component of Teich_b.

DEFINITION: Let M be a hyperkaehler manifold, Teich_b its birational Teichmüller space, and Γ the mapping class group. The quotient Teich_b / Γ is called **the birational moduli space** of M.

Monodromy group and the birational moduli space

THEOREM: Let (M, I) be a hyperkähler manifold, and W a connected component of its birational moduli space. **Then W is isomorphic to $\mathbb{P}\text{er}/\Gamma$, where $\mathbb{P}\text{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ and Γ is an arithmetic group in $O(H^2(M, \mathbb{R}), q)$**, called **the monodromy group**.

REMARK: Γ_I is a group generated by monodromy of the Gauss-Manin local system on $H^2(M)$.

A CAUTION: Usually "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M, \mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, **it is false**.

REMARK: Further on, **I shall freely identify $\mathbb{P}\text{er}$ and Teich_b.**

Ergodicity of the monodromy group action

The moduli space $\mathbb{P}\mathrm{er}/\Gamma_I$ **is extremely non-Hausdorff**.

THEOREM: Let $\mathbb{P}\mathrm{er}$ be a component of a birational Teichmüller space, and Γ its monodromy group. Let $\mathbb{P}\mathrm{er}_e$ be a set of all points $L \subset \mathbb{P}\mathrm{er}$ such that the orbit $\Gamma \cdot L$ is dense (such points are called **ergodic**). **Then $Z := \mathbb{P}\mathrm{er} \setminus \mathbb{P}\mathrm{er}_e$ has measure 0.**

Proof. Step 1: Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. **Then Γ-action on G/H is ergodic,** by Moore's theorem.

Step 2: Ergodic orbits are dense, non-ergodic orbits have measure 0. ∎

REMARK: This implies that **"almost all" Γ-orbits in G/H are dense**.

REMARK: Generic deformation of M has no rational curves, and no non-trivial birational models. Therefore, outside of a measure zero subset, $\mathrm{Teich} = \mathrm{Teich}_b$. This implies that almost all complex structures on M are ergodic.

Construction of ergodic complex structures on K3 surfaces

PROPOSITION: Let I be an ergodic complex structure on a 2-dimensional torus, and let (X, I) be its Kummer surface. **Then (X, I) is ergodic.**

Proof: Let Per be the period space of K3, Γ its mapping class group, and $\operatorname{Per}_K \subset \operatorname{Per}$ the period space of Kummer surfaces (we take a connected component, so that it is identified with the Torelli space of a torus). Shafarevich–Pyatetsky-Shapiro proved that $\Gamma \cdot \operatorname{Per}_K$ is dense in Per.

The mapping class group Γ_K of a torus is embedded to Γ, fixing Per_K, and $\Gamma_K \cdot I \subset \operatorname{Per}_K$ is dense in Per_K, because I is ergodic. Therefore, $\Gamma \cdot I$ is dense in $\Gamma \cdot \operatorname{Per}_K$ which is dense in Per. ■

Construction of ergodic complex structures

DEFINITION: Let M be hyperkähler, and $\eta \in H^2(M,\mathbb{Z})$ be a non-zero class. Denote by Per_η the set of all $l \in \operatorname{Per}$ such that $l \perp \eta$. This is equivalent to η being of type (1,1) with respect to the corresponding complex structure. When $q(\eta,\eta) > 0$, Per_η is called **the polarized birational Teichmüller space**, and classifies manifolds with a given big cohomology class up to birational correspondence.

THEOREM: (Anan′in-V.)
For each η, **the orbit $\Gamma \cdot \operatorname{Per}_\eta$ is dense** in Per.

COROLLARY: Let I be an ergodic complex structure on a K3 surface or a torus X. **Then its Hilbert scheme $M := X^{[n]}$ is ergodic** (for a K3), and **the corresponding generalized Kummer $M := X^{[n]}/X$ is ergodic** when X is a torus.

Proof: The birational Teichmuller space for Hilbert schemes or tori is embedded to $\operatorname{Teich}(M)$ as $\operatorname{Per}_\eta \hookrightarrow \operatorname{Per}(M)$, where $\eta \in H^2(M,\mathbb{Z})$ is cohomology class of the exceptional divisor. Since the orbit of I is dense in $\operatorname{Per}(X)$, it is dense in Per_η, and $\Gamma \operatorname{Per}_\eta$ is dense in Per. ■