Exotic hypercomplex structures on complex tori

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a joint work with Alberto Pipitone Federico.

HYPERCOMPLEX MANIFOLDS

DEFINITION: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$. Suppose that I, J, K are integrable almost complex structures. Then (M, I, J, K) is called **a hypercomplex manifold**.

THEOREM: (M. Obata, 1952)
Let (M, I, J, K) be a hypercomplex manifold. **Then M admits a unique torsion-free affine connection preserving I, J, K.**

REMARK: Converse is also true. Suppose that I, J, K are operators defining quaternionic structure on TM, and ∇ a torsion-free, affine connection preserving I, J, K. **Then I, J, K are integrable almost complex structures, and (M, I, J, K) is hypercomplex.**

Holonomy of Obata connection lies in $GL(n, \mathbb{H})$. Conversely, **a manifold equipped with an affine, torsion-free connection with holonomy in $GL(n, \mathbb{H})$ is hypercomplex**.

This can be used as a definition of a hypercomplex structure: **a hypercomplex manifold** (M, ∇, I, J, K) is a manifold equipped with a torsion-free connection such that its holonomy preserves a quaternionic structure on a tangent bundle.

Exotic hypercomplex structures on hyperkähler manifolds

DEFINITION: A hypercomplex manifold (M, ∇, I, J, K) is called **hyperkähler** if the holonomy $\mathcal{H}\text{ol}(\nabla)$ of ∇ is compact. In this case, $\mathcal{H}\text{ol}(\nabla)$ preserves a quaternionic invariant Riemannian metric g. Such metric is called **hyperkähler**. A **hyperkähler structure** is (M, ∇, I, J, K, g); in this situation, **∇ is the Levi-Civita connection**.

THEOREM: Let (M, I, J, K) be a compact hypercomplex manifold. Assume that (M, I) admits a Kähler structure **Then (M, I) admits a hyperkähler structure (I, J', K').**

DEFINITION: Let (M, I, J, K) be a compact hypercomplex manifold. Assume that (M, I) admits a Kähler structure. The hypercomplex structure (I, J, K) is called **exotic** if it is not compatible with a hyperkähler metric, that is, if the holonomy of its Obata connection is non-compact.

The main result today

THEOREM: (Alberto Pipitone Federico, V.)
There are no exotic hypercomplex structures on a compact torus.

Exotic hypercomplex structures on K3

THEOREM: Exotic hypercomplex structures on K3 do not exist.

Proof. Step 1: Let (M, I, J, K) be a hypercomplex structure on a K3, and Θ the curvature of Obata connection on its canonical bundle $K_{M,I} = K_{M,J} = K_{M,K}$. Since Θ is of type (1,1) for I, J, K, it is $SU(2)$-invariant with respect to the $SU(2)$-action on $\Lambda^*(M)$ generated by quaternions. However, for any $SU(2)$-invariant form Θ, and any Hermitian metric g, one has $\Theta \wedge \Theta = -\|\Theta\|_g^2 \operatorname{Vol}_g$. On the other hand, Θ is exact, because the canonical bundle of a K3 is trivial. **This implies that the Obata connection on the canonical bundle $K_{M,I}$ is flat.** Given that $\pi_1(K3) = 0$, **we obtain that $K_{M,I}$ is trivialized by an Obata-parallel section.**

Step 2: The Obata-parallel sections of the canonical bundle are closed 2-forms (any parallel differential form is closed, if the connection is torsion-free). Varying the complex structure, **we obtain a rank 3 space W of parallel differential forms,** $\omega_I, \omega_J, \omega_K$; the corresponding metric is hyperkähler, because its holonomy belongs to the stabilizer of $\omega_I, \omega_J, \omega_K$, that is, Sp(1). ■

Twistor spaces for hypercomplex manifolds

DEFINITION: **Induced complex structures** on a hypercomplex manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** $\mathrm{Tw}(M)$ of a hypercomplex manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\mathrm{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\mathrm{Tw}} = I_m \oplus I_J : T_x \mathrm{Tw}(M) \to T_x \mathrm{Tw}(M)$ satisfies $I_{\mathrm{Tw}}^2 = -\mathrm{Id}$. **It defines an almost complex structure on $\mathrm{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, then $\mathrm{Tw}(M) = \mathrm{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

REMARK: **For M compact, $\mathrm{Tw}(M)$ never admits a Kähler structure.**

Rational curves on $\operatorname{Tw}(M)$.

DEFINITION: An ample rational curve on a complex manifold M is a smooth curve $S \cong \mathbb{C}P^1 \subset M$ such that $NS = \bigoplus_{k=1}^{n-1} \mathcal{O}(i_k)$, with $i_k > 0$. It is called **a quasiline** if all $i_k = 1$.

THEOREM: ("twistor spaces are rationally connected")
Let M be a compact complex manifold containing a an ample rational line. **red any N points $z_1, ..., z_N$ can be connected by an ample rational curve.**

CLAIM: Let M be a hyperkähler manifold, $\operatorname{Tw}(M) \xrightarrow{\sigma} M$ its twistor space, $m \in M$ a point, and $S_m = \mathbb{C}P^1 \times \{m\}$ the corresponding rational curve in $\operatorname{Tw}(M)$. **Then S_m is a quasiline.**

Proof: Since the claim is essentially infinitesimal, it suffices to check it when M is flat. **Then $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus 2p}) \cong \mathbb{C}P^{2p+1} \backslash \mathbb{C}P^{2p-1}$, and S_m is a section of $\mathcal{O}(1)^{\oplus 2p}$.** ■

The twistor data

Let $\check{\tau}$ denote the central symmetry on $\mathbb{C}P^1$; if we identify $\mathbb{C}P^1$ with imaginary unit quaternions, we have $\check{\tau}(L) = -L$. It is **an anticomplex involution without fixed points.**

DEFINITION: The **twistor data** is a complex manifold Tw equipped with the following structures.

1. **A holomorphic submersion** $\pi :$ Tw $\longrightarrow \mathbb{C}P^1$ **and an anticomplex involution** $\tau :$ Tw $\longrightarrow$ Tw which makes this diagram commutative

$$\begin{array}{ccc} \mathrm{Tw} & \xrightarrow{\tau} & \mathrm{Tw} \\ \pi\downarrow & & \downarrow\pi \\ \mathbb{C}P^1 & \xrightarrow{\check{\tau}} & \mathbb{C}P^1 \end{array}$$

2. **A connected component** Hor **in the set** $\mathrm{Sec}^\tau \subset$ Sec **of** τ**-invariant sections of** π such that for each $S \in$ Hor, the normal bundle to S is $\mathcal{O}(1)^{2n}$ and for each point $x \in$ Tw there exists a unique $S \in$ Hor passing through x.

REMARK: With any twistor space Tw(M) of a hypercomplex manifold, **one associates the twistor data in a natural way:** $\tau(I, m) = (-I, m)$, and Hor(M) the space of all sections S_m taking $I \in \mathbb{C}P^1$ to $(I, m) \in$ Tw(M), where $m \in M$ is a fixed point.

Hypercomplex structures defined in terms of twistor data

THEOREM: (HKLR)
Let M be a hypercomplex manifold. Then **the twistor data on** $\operatorname{Tw}(M)$ **can be used to recover the hypercomplex structure on** M**, which is identified with** Hor**.** Moreoved, **for any twistor data** $(\operatorname{Tw}, \tau, \operatorname{Hor})$**, there exists a hypercomplex structure** (I, J, K) **on** Hor **such that these twistor data are associated with** (I, J, K)**.**

Proof: *N. J. Hitchin, A. Karlhede, U. Lindström, M. Roček,* Hyperkähler metrics and supersymmetry, *Comm. Math. Phys.* **108** *(1987), 535-589.* ■

Complex tori

DEFINITION: A complex torus is a complex manifold M such that its Albanese map $\operatorname{Alb}: M \longrightarrow \frac{H^0(\Omega^1 M)^*}{H^1(M,\mathbb{Z})}$ is an isomorphism.

REMARK: Any Kähler-type complex structure on a manifold diffeomorphic to a torus has this nature; there are **non-Kähler complex structures on a torus,** not well understood yet. These complex structures don't give "complex torus", because the Albanese map for such manifolds is never an isomorphism.

THEOREM: (F. Catanese)
Let $\mathcal{X}$ be a connected, continuous family of complex structures on a manifold M diffeomorphic to a torus. Assume that for some $I \in \mathcal{X}$, the manifold (M, I) is a complex storus. **Then (M, I_1) is a torus for all $I' \in \mathcal{X}$.**

Proof: Fabrizio M.E. Catanese, *Deformation types of real and complex manifolds*, arXiv:math/0111245, Theorem 4.1. ■

Translations and flat structures on complex tori

REMARK: Let $\theta_1, ...\theta_n$ be holomorphic differentials on a complex torus M. Their antiderivatives define a flat affine chart on M; the corresponding flat affine structure on M is canonically defined. **This also defines a holomorphic flat affine connection on M.**

REMARK: Also, each complex torus M is a torsor over the corresponding group manifold, identified with a connected component $\operatorname{Aut}_0(M)$ of $\operatorname{Aut}(M)$, and its action on M is canonically defined. Since $\operatorname{Aut}_0(M)$ is (non-canonically) identified with M, this action is called **the action of the torus on itself by translations**.

Exotic hypercomplex structures on a torus are flat

Theorem 1: Let (I,J,K) be a hypercomplex structure on a complex torus (M,I), and ∇ its Obata connection. **Then ∇ is flat.**

Proof. Step 1: Any anticomplex involution of a torus exchanges holomorphic and antiholomorphic differentials, hence preserves the standard flat structure induced by the complex structure as above. Since the fibers of $\pi: \operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$ are flat, the universal covering $\widetilde{\operatorname{Tw}}(M)$ is an affine bundle, and the anticomplex involution preserves the affine structure. Fixing a horizontal section, **we identify $\widetilde{\operatorname{Tw}}(M)$ with $\operatorname{Tot}(\mathcal{O}(1)^{2n})$; the anticomplex involution also preserves the vector bundle structure.**

Step 2: Since the hypercomplex structure on $\operatorname{Tot}(\mathcal{O}(1)^{2n}) = \widetilde{\operatorname{Tw}}(M)$ is compatible with the vector space structure on fibers, it gives a hypercomplex structure, compatible with the vector bundle operation (addition and multiplication). Such a hypercomplex structure is translation-invariant, hence flat. **We obtain that (M,I,J,K) is a quotient of a flat hypercomplex manifold $\mathbb{H}^n$ by an affine action of $\mathbb{Z}^{4n}$.** ■

REMARK: If the monodromy of Obata connection on M is trivial (or just compact), **it would immediately follow that M is a hyperkähler torus.**

In the rest of this talk, I would discuss **torsion-free flat connections on complex tori.**

Flat affine structures and the development map

DEFINITION: A **flat affine structure** on a manifold M is a flat torsion-free connection.

DEFINITION: Let M be a simply connected flat affine manifold, and $\theta_1, ..., \theta_n \in \Lambda^1 M$ a basis of parallel 1-forms. Since a parallel 1-form is closed and $H^1(M, \mathbb{R}) = 0$, the forms θ_i are exact. Then $\theta_i = dx_i$. The map $\delta : M \to \mathbb{R}^n$ taking m to $(x_1(m), ..., x_n(m))$ is called **the development map.** We consider $\mathbb{R}^n$ as a flat affine manifold, with the standard flat affine structure.

CLAIM: The development map $\delta : M \to \mathbb{R}^n$ **is compatible with the flat affine connections.**

Proof: It takes the coordinate 1-forms $dx_1, ..., dx_n \in \Lambda^1(M)$ to $\theta_1, ..., \theta_n \in \Lambda^1 M$. However, these 1-forms are parallel. ■

Linear and affine holonomy

DEFINITION: Linear holonomy (or **holonomy**) of a flat affine connection ∇ is its monodromy in TM; by definition, the holonomy group belongs to $GL(T_xM)$, where $x \in M$ is a base point.

DEFINITION: Let Aff$(\mathbb{R}^n)$ denote the group of affine transforms of $\mathbb{R}^n$. Clearly, Aff$(\mathbb{R}^n)$ is a semidirect product, Aff$(\mathbb{R}^n) = GL(n,\mathbb{R}) \rtimes \mathbb{R}^n$. The natural map Aff$(\mathbb{R}^n) \longrightarrow GL(n,\mathbb{R})$ is called **the linearization.**

DEFINITION: Let M be a flat affine n-manifold, Aff$(\mathbb{R}^n)$ $\tilde M$ its universal cover $\delta: \tilde M \to \mathbb{R}^n$ the development map, and $a: \pi_1(M) \longrightarrow$ Aff$(\mathbb{R}^n)$ the map taking $\gamma \in \pi_1(M)$ to an element of Aff$(\mathbb{R}^n)$ making the following diagram commutative:

$$\begin{array}{ccc} \tilde M & \stackrel{\delta}{\longrightarrow} & \mathbb{R}^n \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle a} \\ \tilde M & \stackrel{\delta}{\longrightarrow} & \mathbb{R}^n. \end{array}$$

The map $a: \pi_1(M) \longrightarrow$ Aff$(\mathbb{R}^n)$ is called **the affine holonomy map.**

REMARK: The linear holonomy of a manifold **is the linearization if its affine holonomy.**

Non-standard flat affine structures on a torus

REMARK: A flat affine structure on a torus is called **standard** if its linear holonomy is trivial.

Remark 1: Let (M, ∇) be a flat affine torus with the standard flat affine structure. Then $\pi_1(M)$ acts on $\tilde{M}$ by translations, hence **$\tilde{M} = \mathbb{R}^n$ and M is isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$ with the standard flat affine structure.**

REMARK: In *Sullivan, Dennis; Thurston, William Manifolds with canonical coordinate charts: some examples. Enseign. Math. (2) 29 (1983), no. 1-2, 15-25.* Thurston and Sullivan gave examples of non-standard flat affine structures on a torus.

EXAMPLE: Consider the quotient $M := \frac{\mathbb{R}^2 \backslash 0}{\mathbb{Z}}$, where $\mathbb{Z}$ acts by homotheties. **Clearly, the holonomy of M is $\mathbb{Z}$ acting on TM by homotheties.**

EXAMPLE: Consider $\mathbb{Z}^2$-action ρ on $\mathbb{R}^2$ generated by $(x, y) \to (x + 1, y)$ and $(x, y) \to (x + y, y + 1)$. The projection to the second component maps $\frac{\mathbb{R}^2}{\operatorname{im} \rho}$ to S^1, with the fiber S^1, hence $\frac{\mathbb{R}^2}{\operatorname{im} \rho}$ is a torus; its (linear) holonomy is generated by $A(x, y) := (x + y, y)$.

Frid-Goldman-Hirsch theorem

DEFINITION: A flat affine manifold (M, ∇) is called **complete** if $M = \frac{\mathbb{R}^n}{\Gamma}$, where $\Gamma = \pi_1(M)$, with its action factorized through $\mathrm{Aff}(\mathbb{R}^n)$.

CONJECTURE: ("Marcus conjecture") A compact flat affine manifold **is complete if and only if it admits a parallel volume form.**

THEOREM: Let (M, ∇) be a compact flat affine manifold with affine holonomy group nilpotent. **Then the following are equivalent:**

(a) (M, ∇) is complete,

(b) (M, ∇) admits a paralell volume form, and

(c) its linear holonomy action is unipotent.

Proof: Theorem A in *Fried, D., Goldman, W., Hirsch, M.W., Affine manifolds with nilpotent holonomy, Commentarii Mathematici Helvetici 56, 487-523 (1981),* `https://doi.org/10.1007/BF02566225` ■

Frid-Goldman-Hirsch theorem for Obata connection a torus

COROLLARY: Let $W := \mathbb{H}^n$, (M, I, J, K) an exotic hypercomplex structure on a torus, and ∇ its Obata connection. **Then (M, ∇) satisfies (a)-(c) of Frid-Goldman-Hirsch theorem.**

Proof: Since (M, I) is Kähler, it is HKT; since its canonical bundle is trivial and (M, I, J, K) is HKT, the Obata holonomy is contained in $SL(n, \mathbb{H})$ and ∇ fixes a volume form, as shown in *M. Verbitsky, Hyperkähler manifolds with torsion, supersymmetry and Hodge theory, Asian J. of Math., Vol. 6 (4), December 2002).* ■

REMARK: Most importantly, this implies that the flat Obata connection on an exotic hypercomplex torus **is complete.**

Complete flat affie connections on a torus

THEOREM 1: Let (M,∇) be a complete flat affine structure on a compact torus. Then

(i) **its linear holonomy is unipotent.**

(ii) For some real basis $t_1,...,t_n$ in $\mathbb{R}^n$, the action of $\pi_1(M)$ on $\mathbb{R}^n = \widetilde{M}$ is generated by $\tau_1,...,\tau_n$, with $\tau_i(x) := t_i + L_i(x)$. Here $L_1,...,L_n \in GL(n,\mathbb{R})$ is a collection of commuting unipotent matrices which satisfy

$$(L_i - \mathrm{Id})(t_j) = (L_j - \mathrm{Id})(t_i) \quad (*)$$

for any i,j.

(iii) For any collection of commuting affine maps $\tau_1,...,\tau_n$ with $\tau_i(x) := t_i + L_i(x)$, where $t_1,...,t_n$ is a basis and all L_i unipotent and satisfying (*), **there exists a flat affine structure on a torus M with $\tau_1,...,\tau_n$ generating the action of $\pi_1(M)$ on $\mathbb{R}^n$.**

Proof. Step 1: The holonomy of ∇ is unipotent by Fried-Goldman-Hirsch.

Step 2: To prove (ii), we write the generators of $\Gamma = \pi_1(M)$ as $\tau_1,...,\tau_n$, with $\tau_i(x) := L_i(x) + t_i$, where $L_1,...,L_n$ are linear. Clearly, L_i are commuting and (by Fried-Goldman-Hirsch) unipotent. The equation (*) follows because $\tau_i\tau_j(x) = L_i(t_j) + L_iL_j(x) + t_i$. To complete the proof of (ii), **it remains to show that all the t_i are linearly independent.**

Step 3: Linear independence of t_i is clear because **expotents of $G := \sum \alpha_i \log \tau_i(x)$ define a free $\mathbb{R}^n$-action on $\mathbb{R}^n$;** this also proves (iii), because G is a a free commutative Lie group acting on $\mathbb{R}^n$. ■

Exotic hypercomplex structures on tori do not exist

REMARK: In assumptions of Theorem 1, let Ψ be the tensor written in the basis $t_1, ..., t_n$ as

$$\Psi(u,v) := \sum_{i,j,k=1}^{2n} (A_i)_j^k u^i v^j t_k,$$

where $A_i = L_i - \mathrm{Id}$. **Then (*) can be rewritten as $\Psi \in \mathrm{Sym}^2 V^* \otimes V$.** If, in addition, the flat connection preserves a complex or a hypercomplex structure on a torus, this would mean that $A_i \in GL(n, \mathbb{C})$ or $A_i \in GL(n, \mathbb{H})$.

THEOREM: Exotic hypercomplex structures on tori do not exist.

Proof: Let (M, I, J, K) be a hypercomplex structure and ∇ the Obata connection on a complex torus. Then ∇ is flat and complete, hence Theorem 1 can be applied. Let $\Psi \in \mathrm{Sym}^2(V) \otimes V^*$ be the tensor defined above; by construction, Ψ commutes commutes with I, J, K on last two arguments. This forces Ψ to be zero, since

$$K\Psi(x,y) = \Psi(IJx, y) = \Psi(Jx, Iy) = \Psi(x, JIy) = -K\Psi(x,y).$$

■