Octonions and G_2-manifolds

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Holonomy group

DEFINITION: (Cartan, 1923) Let (B,∇) be a vector bundle with connection over M. For each loop γ based in $x\in M$, let $V_{\gamma,\nabla}: B|_x \to B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B,∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: A bundle is **flat** (has vanishing curvature) **if and only if its restricted holonomy vanishes.** Indeed, **the connected component of the holonomy group of a bundle (B,∇) is a Lie group, with the Lie algebra generated by all curvature elements of M transported to a given point by the connection** (Ambrose, Singer).

REMARK: If $\nabla(\varphi)=0$ for some tensor $\varphi\in B^{\otimes i}\otimes(B^*)^{\otimes j}$, **the holonomy group preserves φ.**

DEFINITION: **Holonomy of a Riemannian manifold** is holonomy of its Levi-Civita connection.

EXAMPLE: Holonomy of a Riemannian manifold lies in $O(T_xM, g|_x)=O(n)$.

EXAMPLE: Holonomy of a Kähler manifold lies in $U(T_xM, g|_x, I|_x)=U(n)$.

The Berger's list

THEOREM: (Berger's theorem, 1955)
Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

REMARK: If the holonomy group is not irreducible, the manifold M **locally splits onto a product of Riemannian manifolds with irreducible holonomy** (de Rham). **This splitting is global,** if M is complete and simply connected.

$GL(n,\mathbb{R})$ acting on $\Lambda^k(\mathbb{R}^n)$ with an open orbit

THEOREM: Consider the action of $GL(n,\mathbb{R})$ on $\Lambda^k(\mathbb{R}^n)$. **This action has open orbit in the following cases: when $k=0,1,2,n-2,n-1,n$ and when $k=3,n-3$ and $n=6,7,8$.**

Proof for some special cases: It is clear that $GL(n,\mathbb{R})$-action has an open orbit when $k=0,1,2,n-2,n-1,n$: non-degenerate 2-forms are all conjugate, and non-degeneracy is an open condition. In the case $k=3,n-3$ and $n>8$ and $k>3$, $n\geqslant 8$, the $GL(n,\mathbb{R})$-action cannot have an open orbit, which follows from a dimension count.

Existence of an open orbit for $n=6,7,8$ and $k=3$ is a non-trivial exercise, especially when $n=8$ (G. B. Gurevich, "Classification of trivectors of rank eight," Dokl. Akad. Nauk SSSR 2 (1935), 353-355.) **Today we focus on $n=7$, $k=3$, and prove the existence of an open orbit.**

The Fano projective plane

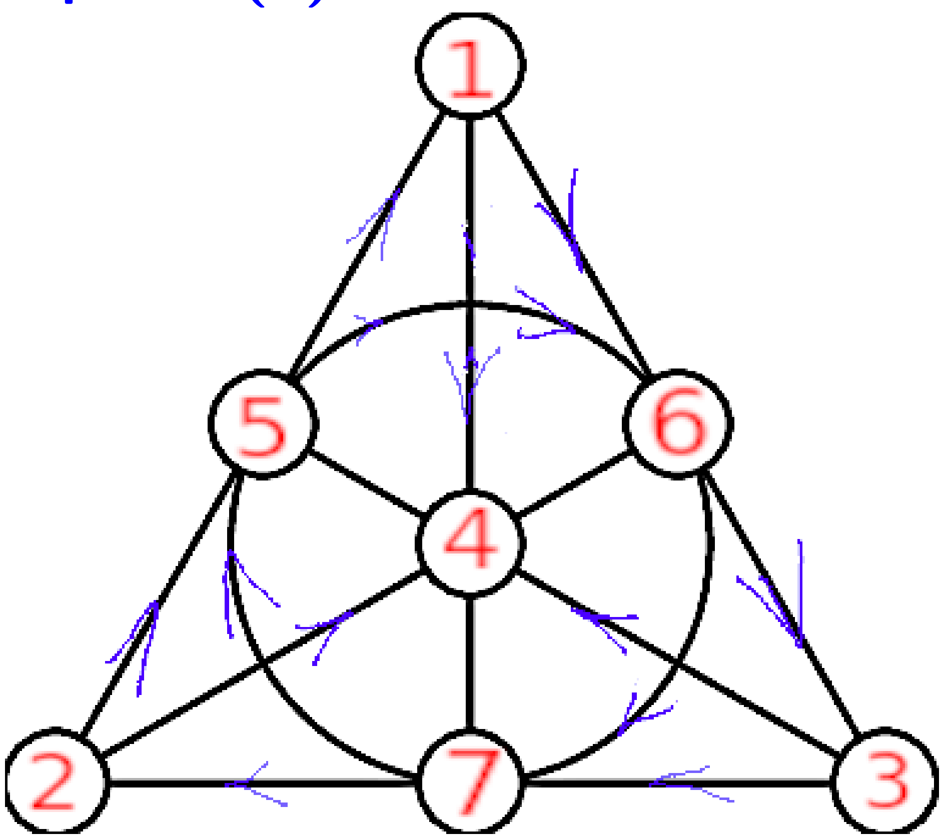
CLAIM: $PGL(\mathbb{F}_2, 2) = \mathbb{Z}/3$.

Proof: Indeed, $GL(\mathbb{F}_2, 2)$ has cardinality 6 (there are 3 choices for an image of the first vector of a basis, 2 for the second), hence $PGL(\mathbb{F}_2, 2) = GL(\mathbb{F}_2, 2)/(\mathbb{Z}/2)$ has cardinality 3. ■

REMARK: This implies that **all lines on $\mathbb{F}_2\mathbb{P}^n$ have circle ordering.**

DEFINITION: The Fano projective plane is $\mathbb{F}_2\mathbb{P}^2$, equipped with the $PGL(\mathbb{F}_2, 3)$-action. The lines (each of which has 3 points) are circle oriented, and the $PGL(\mathbb{F}_2, 3)$ preserves the orientation:

The Fano projective plane (2)



PROPOSITION: Let $V = \mathbb{R}^7$. Choose the basis $v_1, ..., v_7 \in V$, and let $v_{ijk} := v_i \wedge v_j \wedge v_k \in \Lambda^3(\mathbb{R}^7)$. Consider the form

$$\eta := v_{125} + v_{345} - v_{136} + v_{246} + v_{147} + v_{237} + v_{567},$$

and let $PGL(\mathbb{F}_2, 3)$ act on v_i as on the vertices of the Fano plane. **Then $PGL(\mathbb{F}_2, 3)$ preserves η.**

Proof: Indeed, **the lines of the Fano plane correspond to the monomials terms in η, and their sign to the circle orientation.** ■

3-forms on $\mathbb{R}^7$ and scalar product

THEOREM: Let $V = \mathbb{R}^7$. The form $\eta \in \Lambda^3(V)$ defines the following scalar $\Lambda^7(V)$-valued product on V^*: $g_\eta(x,y) = i_x\eta \wedge i_y\eta \wedge \eta$. Consider the form

$$\eta := v_{125} + v_{345} - v_{136} + v_{246} + v_{147} + v_{237} + v_{567}$$

defined as above. **Then the corresponding scalar product g_η is can be written as $g_\eta(e_i, e_j) = \delta_{ij}\operatorname{Vol}_V$,** where e_i is the dual basis in V^*.

Proof: The group of $PGL(\mathbb{F}_2, 3)$ automorphisms of the Fano projective plane $\mathbb{F}_2\mathbb{P}^2$ acts transitively on $\{v_1, ..., v_7\}$, preserving this form, and acts transitively on the set of pairs $v_i \neq v_j$, hence it suffices to check that $g_\eta(v_1, v_1) = 1$ and $g_\eta(v_1, v_2) = 0$. This is a trivial calculation. ■

REMARK: Since $\eta \in \operatorname{Sym}^2(V) \otimes \Lambda^7(V)$, its determinant $\det(g_\eta)$ is an element in $\Lambda^7(V)^{\otimes 2} \otimes (\Lambda^7(V)^{\otimes 7} = (\Lambda^7(V)^{\otimes 9}$. Then $\sqrt[9]{\det(g_\eta)}$ is a volume form on V. Denote this volume form by Vol_η. Then $\frac{g_\eta}{\operatorname{Vol}_\eta}$ is non-degenerate scalar product on V^*. Denote by g the dual scalar product on V; **this scalar product is positive definite and unambiguously determined by η, hence preserved by its stabilizer $G_\eta \subset GL(7, \mathbb{R})$.**

COROLLARY: Let $V = \mathbb{R}^7$, η the 3-form defined above, and $G_\eta \subset GL(7, \mathbb{R})$ its stabilizer. **Then G_η is compact, and preserves a Euclidean form g unambiguously determined by ρ.** ■

Vector product

DEFINITION: Let V be a vector space equipped with a non-degenerate scalar product g and a 3-form $\eta \in \Lambda^3(V^*)$. Define **the vector product** $V \times V \to V$ as $x, y \mapsto g^{-1}(i_y i_x(\eta))$, where $g^{-1}: V^* \to V$ is the natural isomorphism induced by g.

CLAIM: Let $V = \mathbb{R}^7$ and $\eta \in \Lambda^3(V^*)$ the 3-form defined above. Then **the vector product $v_i \times v_j$ is $\pm v_k$, where v_k is the third node on the line connecting v_i and v_j on the Fano plane, and the sign is determined by the orientation.**

Proof: $i_{v_i} i_{v_j} \eta = v_k^*$, where $\{v_i^*\}$ is the dual basis. ∎

The octonions

REMARK: From now on, **we use g to identify V and V^*, and consider η as an element of $\Lambda^3(V) = \Lambda^3(V^*)$.**

DEFINITION: Let $V = \mathbb{R}^7$ and $\eta \in \Lambda^3(V)$ the 3-form defined above. In these assumptions, we define **the octonion algebra** as $\mathbb{O} := \mathbb{R} \cdot 1 \oplus V$, with $\operatorname{Im} \mathbb{O} = V$ ("the imaginary octonions") and $\operatorname{Re} \mathbb{O} = \mathbb{R}$ ("the real octonions"), and the octonion product defined as follows: for any real octonion α and any octonion β, we have $\alpha \cdot \beta = \alpha\beta$, and for two imaginary octonions u, v, the product $u \cdot v := -g(u,v) \cdot 1 + u \times v$, where $\times$ is the vector product on V.

REMARK: For three basis vectors v_i, v_j, v_k such that $v_i \times v_j = v_k$, the algebra $\mathbb{R}1 \oplus \langle v_i, v_j, v_k \rangle$ **is isomorphic to quaternions.**

REMARK: **The octonion algebra is non-associative!** However, **it is a division algebra**, as we shall see today.

REMARK: Frobenius (1877) **proved that any division algebra over reals is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.**

The group G_2 and octonions

DEFINITION: Let $V = \mathbb{R}^7$ and $\eta \in \Lambda^3(V)$ the 3-form defined above. **Define** $G_2 \subset GL(7,\mathbb{R})$ as the group of automorphisms of V preserving η; in other words, $G_2 = \operatorname{Aut}(\mathbb{O})$.

Proposition 1: $\dim_{\mathbb{R}} G_2 \leqslant 14$.

Proof: Take the standard basis $\{v_i\} \subset V$, and let $\varphi \in G_2$. By definition, the action of φ on V is compatible with the vector product. The space $\langle v_1, v_2, v_5\rangle$ is isomorphic to imaginary quaternions with the standard vector product. Then $\varphi(v_1)$ belongs to a unit sphere $S(V) \cong S^6$, and $\varphi(v_2)$ in $\varphi(v_1)^\perp \cap S(V) = S^5$. The image of $v_5 = v_1 \times v_2$ is determined uniquely via $\varphi(v_5) = \varphi(v_1) \times \varphi(v_2)$. The image of v_3 belongs to $\langle \varphi(v_1), \varphi(v_2), \varphi(v_5)\rangle^\perp \cap S(V) \cong S^3$, which is 3-dimensional, and the vector products of v_1, v_2, v_3 generate the rest of v_i, hence **φ is uniquely determined by $(\varphi(v_1), \varphi(v_2), \varphi(v_3)) \in S^6 \times S^5 \times S^3$, which is 14-dimensional.** ■

REMARK: In the next slide, **we prove that $\dim G_2 = 14$.**

The orbit of η in $\Lambda^3(\mathbb{R}^7)$ is open

THEOREM: **$GL(7,\mathbb{R})$ acts on $\Lambda^3(\mathbb{R}^7)$ with an open orbit.**

Proof. Step 1: Let $\eta \in \Lambda^3(\mathbb{R}^7)$ be the 3-form defined above, and $G_\eta \subset GL(7,\mathbb{R})$ its stabilizer. Sinc $\dim \Lambda^3(\mathbb{R}^7) = 35$, $\dim GL(7,\mathbb{R}) = 49$, the dimension of an orbit $GL(7,\mathbb{R})\cdot\eta$ is $49 - \dim G \leqslant 35$, which gives $\dim G \geqslant 49-35 = 14$, **with equality occuring when the orbit is open.**

Step 2: $\dim G \leqslant 49 - 35 = 14$ (Proposition 1). ■

COROLLARY: $\dim G_2 = 14$. ■

COROLLARY: The group G_2 **acts transitively on the unit sphere** $S^6 \subset V$.

Proof. Step 1: We start by proving that the orbit $G_2 \cdot v_1 \subset S^6$ is 6-dimensional. Consider the standard 3-form $\eta \in \Lambda^3(V)$, where $V = \mathbb{R}^7$. An element $\varphi \in G_2$ is determined by the image of v_1, v_2, v_3, with $\varphi(v_2) \in S^6 \cap \varphi(v_1)^\perp = S^5$ and $\varphi(v_2) \in S^6 \cap \langle \varphi(v_1), \varphi(v_2), \varphi(v_5)\rangle^\perp = S^3$. Therefore, $14 = \dim G_2 \leqslant \dim G_2 \cdot v_1 + 5 + 3$, which implies that $6 = 14 - 8 \leqslant \dim G_2 \cdot v_1$.

Step 2: Since the orbit is a homogeneous space, it is actually a 6-dimensional submanifold in S^6. **However, G_2 is compact, hence any its orbit is also compact.** A compact 6-dimensional submanifold of S^6 is S^6 itself. ■

Octonions are left alternative algebra

DEFINITION: We say that a non-associative algebra **is left alternative** if it satisfies $x(xy) = (xx)y$ for all x, y. **This property is a weakened form of associativity.**

CLAIM: The octonion algebra is left alternative. In particular, **for any $x \in \operatorname{Im}\mathbb{O}$ and any $y \in \mathbb{O}$, we have $x^2 = -|x|^2$, and $(x(xy)) = -|x|^2 y$.**

Proof. Step 1: Left alternativity is implied by the second assertion. Indeed, if $\lambda \in \operatorname{Re}\mathbb{O}$, we have $(\lambda + x)(\lambda + x)y = (xx)y + \lambda xy + \lambda^2 y$ and $(\lambda + x)((\lambda + x)y) = x(xy) + \lambda xy + \lambda^2 y$.

Step 2: Since G_2 acts transitively on the sphere and commutes with octonion product, it suffices to show this statement when $x = v_1$. In this case, $v_1(v_1 v_i) = (v_1^2)v_i = -v_i$, as follows from the multiplication table on the basis vectors encoded by the Fano plane. ■

Octonions are division algebra

DEFINITION: Given $x \in \mathbb{O}$, define $\overline{x} := \operatorname{Re} x - \operatorname{Im} x$. This operation is called **conjugation**, or **octonion conjugation**.

CLAIM: Similarly to quaternions, **the octonion conjugation has the following properties:** $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$ and $x\overline{x} = |x|^2$.

Proof: It suffices to check $\overline{(xy)} = \overline{y} \cdot \overline{x}$ when $x, y \in \operatorname{Im} \mathbb{O}$, where it follows because $a \times b = -b \times a$. The second assertion is implied by $v_i \times v_j = -v_j \times v_i$, hence

$$\left(\lambda + \sum_{i=1}^{7} a_i v_i\right)\left(\lambda - \sum_{i=1}^{7} a_i v_i\right) = \lambda^2 - \sum_{i=1}^{7} |a_i|^2 v_i^2 = \lambda^2 + \sum_{i=1}^{7} |a_i|^2.$$

■

THEOREM: The octonion algebra $\mathbb{O}$ is a division algebra (that is, $\mathbb{O}$ has no zero divisors).

Proof. Step 1: From left alternativity **it follows immediately that $\overline{x}(xy) = (\overline{x}x)y) = |x|^2 y$.** Indeed, let $x = a + b$, whjere $a = \operatorname{Re} x$ and $b = \operatorname{Im} x$, Then $\overline{x}(xy) = (a - b)((a + b)y) = aby - bay + a^2 y - b^2 y = (|a|^2 + |b|^2)y$.

Step 2: if $xy = 0$, then $\overline{x}(xy) = (\overline{x}x)y = |x|^2 y = 0$, hence either $x = 0$ or $y = 0$. ■

G_2 acts transitively on pairs of vectors

PROPOSITION: Let $V = \mathbb{R}^7$ be the space with standard action of G_2. **Then G_2 acts transitively on the set X pairs (x, y) of orthogonal vectors with $|x| = |y| = 1$.**

Proof: Clearly, X is 11-dimensional. Since G_2 is compact, **it suffices to show that the orbits $G_2 \cdot (v_1, v_2)$ of G_2-action on X are also 11-dimensional.**

Step 2: Any element $\varphi \in G_2$ is determined by $\varphi(v_1), \varphi(v_2), \varphi(v_3)$; if $\varphi(v_1) = v_1, \varphi(v_2) = v_2$, then $\varphi(v_3)$ is a point on the 3-sphere $S^6 \cap \langle \varphi(v_1), \varphi(v_2), \varphi(v_5) \rangle^\perp$, hence the dimension of the stabilizer of the pair $(v_1, v_2) \in X$ is at most 3-dimensional, and the orbit is at least 11-dimensional. ■

COROLLARY: Any two non-collinear imaginary octonions **generate a quaternion subalgebra $\mathbb{H} \subset \mathbb{O}$.** ■

Stable 3-forms on 7-manifolds

DEFINITION: A 3-form on $\mathbb{R}^7$ is **stable** if its stabilizer in $GL(7,\mathbb{R})$ is isomorphic to G_2, Let $\eta \in \Lambda^3(M)$ be a 3-form on a 7-manifold M. The form η is called **stable** if the stabilizer of $\eta\big|_{T_xM}$ is isomorphic to G_2 for each $x \in M$.

REMARK: This is an open property: a small deformation of a stable form ρ is stable. Indeed, the $GL(7,\mathbb{R})$-orbit of ρ is open, and all forms in this orbit are also stable.

REMARK: Any stable 3-form ρ on M defines a $\Lambda^7(M)$-valued metric $g_\rho(x,y) := i_x\rho \wedge i_y\rho \wedge \rho$ on TM, and after the natural trivialization of the line bundle $\Lambda^7(M)$, it defines the Riemannianmetric g on M.

DEFINITION: Let $\rho \in \Lambda^3(M)$ be a stable 3-form. The pair (M,ρ) is called **a holonomy G_2-manifold** if $\nabla(\rho) = 0$ where ∇ denotes the Levi-Civita connection.

EXAMPLE: A 7-torus with the standard 3-form is a holonomy G_2-manifold, because ρ is invariant under the parallel transport, hence the connection form is constant, which implies that $\nabla(\rho) = 0$.

G_2 and $SU(3)$

PROPOSITION: Let (V, η) be a 7-dimensional vector space with the standard 3-form, and $x \in V$ a non-zero vector. **Then the stabilizer of x in G_2 is isomorphic to $SU(3)$.**

Proof. Step 1: Let G_x be the stabilizer of x in G_2. Then $G_x \subset O(x^\perp)$. Also, G_x preserves the complex structure $I(v) := x \times v$, where $\times$ is the vector product on V, hence $G_x \subset U(3)$. Since $\dim G_x = \dim G_2 - \dim S^6 = 8$, and $\dim U(3) = 9$, it is a codimension 1 subgroup. **It is not hard to prove that $U(3)$ contains only one connected codimension 1 subgroup, using the Lie algebras,** but we will use a more explicit argument.

Step 2: Consider the 3-form $\eta\big|_{x^\perp}$. **If we prove that this form is non-zero and of Hodge type (3,0)+(0,3), we will obtain that $G_x \subset SU(3)$,** and the equality $G_x = SU(3)$ follows because $\dim G_x = \dim SU(3)$.

Step 3: $\rho\big|_{x^\perp} \neq 0$ because **for $x = v_1$ it is equal to $v_{345} + v_{246} + v_{237} + v_{567}$.**

Step 4: The following 3 vectors in $v_1^\perp$ have type (1,0): $z_1 = v_2 + \sqrt{-1}v_5, z_2 = v_4 + \sqrt{-1}\, v_7, z_3 = v_6 + \sqrt{-1}\, v_3$, hence the real part of the (3,0)-vector is $v_{246} - v_{273} - v_{543} - v_{657} = v_{246} + v_{237} + v_{345} + v_{567}$. ■

G_2-manifolds and Calabi-Yau manifolds

COROLLARY: Let (W, I) be a 6-dimensional vector space, I a complex structure operator, $z_1 = v_2 + \sqrt{-1}\, v_5, z_2 = v_4 + \sqrt{-1}\, v_7, z_3 = v_6 + \sqrt{-1}\, v_3$ the basis in $W^{1,0}$, and $\Omega := z_1 \wedge z_2 \wedge z_3$. Denote by ω the standard Hermitian form on W, $\omega = v_2 \wedge v_5 + v_4 \wedge v_7 + v_6 \wedge v_3$. Consider the space $V := \mathbb{R} \cdot t \oplus W$, and let $\rho := \operatorname{Re} \Omega + t \wedge \omega$. **Then ρ is a stable 3-form.**

DEFINITION: A Calabi-Yau manifold is a Riemannian 6-manifold with holonomy group in $SU(3)$. In other words, **a manifold (X, I, g, Ω) is Calabi-Yau if I is a parallel almost complex structure, and Ω a parallel, non-zero 3-form.**

COROLLARY: Let (X, I, g, Ω) be an almost complex Hermitian 6-manifold equipped with a $(3,0)$-form Ω which satisfies $|\operatorname{Re} \Omega| = 4$. Consider the manifold $M = X \times I$, where I is a 1-dimensional oriented Riemannian manifold with dt unit 1-form, and let $\rho := \omega \wedge dt + \operatorname{Re} \Omega \in \Lambda^3 M$. **Then ρ is stable, and the corresponding Riemannian form $g \in \operatorname{Sym}^2 T^* M$ is the product form.** Moreover, **(M, ρ) is a holonomy G_2-manifold if and only if (X, I, g, Ω) is Calabi-Yau.**

Proof: By Corollary 2, ρ is a stable 3-form, which is parallel with respect to Levi-Civita, because the forms ω and $\operatorname{Re}\Omega$ are parallel with respect to the Levi-Civita. ■

For further reading, see "Stable forms and special metrics" (Nigel Hitchin,`https://arxiv.org/abs/math/0107101`*), "The Octonions" (John C. Baez,* `https://math.ucr.edu/home/baez/octonions/octonions.html`*), and "Riemannian Holonomy Groups and Calibrated Geometry" by Dominic Joyce (Oxford Graduate Texts in Math)*.