Formally Kähler structure on a knot space of a G_2-manifold.

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Geometric structures in mathematical physics

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Motivation

Quaternionic/hyperkähler/hypercomplex manifolds have twistor spaces. All the structures of quaternionic geometry are interpreted as holomorphic structures on these twistor spaces. **We want to interpret G_2-geometry in a similar fashion**.

To define a twistor space of a G_2-manifold, **it is necessary to sacrifice something.**

It turns out that **sacrificing finite-dimensionality suffices.**

Knot spaces

DEFINITION: Let M be a smooth manifold. A **knot space** on M is a space of **non-parametrized, immersed, oriented loops**, represented by a map which is injective outside a finite set.

DEFINITION: A **Fréchet space** is an infinite-dimensional topological vector space V admitting a translation-invariant complete metric.

EXAMPLE: The space of smooth functions on a manifold is a Fréchet space.

DEFINITION: **A differentiable map** of Fréchet spaces is a map which can be approximated at each point by a continuous linear map, up to a term which decays faster than linear, in the sense of this metric.

DEFINITION: A **Fréchet manifold** is a ringed space, locally modeled on a space of differentiable functions on a Fréchet space

EXAMPLE: A group of diffeomorphisms is a Fréchet Lie group.

EXAMPLE: The knot space is a Fréchet manifold.

Formally Kähler manifolds

DEFINITION: Let F be a Fréchet manifold. The **sheaf of vector fields** TF on F is a sheaf of continuous derivations of its structure sheaf.

REMARK: A commutator of two derivations is again a derivation. Therefore, **TF is a sheaf of Lie algebras.**

DEFINITION: Let F be a Fréchet manifold, and $I: TF \longrightarrow TF$ a smooth $C^\infty F$-linear endomorphism of the tangent bundle satisfying $I^2 = -1$. Then I is called **an almost complex structure on F**.

REMARK: Clearly, I defines a decomposition $TF \otimes \mathbb{C} = T^{1,0}F \oplus T^{0,1}F$, where $T^{1,0}F$ is the $\sqrt{-1}$-eigenspace of I, and $T^{0,1}F$ the $-\sqrt{-1}$-eigenspace. Indeed, $x = \frac{1}{2}(x + \sqrt{-1}\,Ix) + \frac{1}{2}(x - \sqrt{-1}\,Ix)$.

DEFINITION: An almost complex structure on a Fréchet manifold (F, I) is called **formally integrable**, if $[T^{1,0}F, T^{1,0}F] \subset T^{1,0}F$,

DEFINITION: Let (F, I) be a formally integrable almost complex Fréchet manifold, g a Hermitian structure on F, and ω be the corresponding $(1,1)$-form. We say that (F, I, g) is **formally Kähler** if ω is closed.

Knot spaces of Riemannian 3-manifolds

J. L. Brylinski, *The Kähler geometry of the space of knots in a smooth threefold,* Preprint, Penn. State Univ., University Park, PA, 1990

J. L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization,* Progr. Math., vol. 107, Birkhäuser Boston, Boston, MA, 1993.

LeBrun, Claude, *A Kähler structure on the space of string worldsheets*, Classical Quantum Gravity 10 (1993), no. 9, L141–L148.

Lempert, László, *Loop spaces as complex manifolds,* J. Differential Geom. 38 (1993), no. 3, 519–543.

DEFINITION: Let Knot(M) be the space of knots on a Riemannian 3-manifold M. For each $S \in$ Knot(M), T_SKnot(M) is the space of sections of a normal bundle NS. Let γ be a unit tangent vector to S. **The vector product with γ defines a complex structure on the vector space NS.**

THEOREM: (Brylinski) **This complex structure is formally integrable**. Moreover, **the standard metric on $Knot(M)$ is formally Kähler.**

G_2-manifolds

DEFINITION: Let $\rho \in \Lambda^2\mathbb{R}^7$ be a 3-form on $\mathbb{R}^7$. We say that ρ is **non-degenerate** if the dimension of its stabilizer is maximal:

$$\dim St_{GL(7)}\rho = \dim GL(7) - \dim \Lambda^3(\mathbb{R}^7) = 49 - 35 = 14.$$

In this case, $St(\rho)$ is one of two real forms of a 14-dimensional Lie group $G_2(\mathbb{C})$. We say that ρ is **non-split** if it satisfies $St(\rho|_x) \cong G_2$, where G_2 denotes the compact real form of $G_2(\mathbb{C})$. **A G_2-structure** on a 7-manifold is a 3-form $\rho \in \Lambda^3(M)$, which is non-degenerate and non-split at each point $x \in M$ ("stable", in the sense of Hitchin).

REMARK: A form ρ defines a $\Lambda^7 M$-valued metric on M:

$$g(x,y) = (\rho \lrcorner x) \wedge (\rho \lrcorner y) \wedge \rho$$

This defines a conformal structure on M. The conformal factor is fixed if we want $|\rho| = 1$. Therefore, **every G_2-manifold is equipped with a natural Riemannian structure.**

DEFINITION: An G_2-manifold is called **a holonomy G_2-manifold** if ρ is preserved by the corresponding Levi-Civita connection.

The vector product on a G_2-manifold

DEFINITION: Let $V = \mathbb{R}^7$ be a 7-dimensional real space equipped with a 3-form ρ with $St_{GL(7)}(\rho) = G_2$, and g the G_2-invariant metric defined as above. Define the **octonion vector product** as $x \star y = \rho(x, y, \cdot)^\sharp$. Here $\rho(x, y, \cdot)$ is a 1-form obtained by contraction, and $\rho(x, y, \cdot)^\sharp$ its dual vector.

CLAIM: For each unit vector $x \in V$, the vector product $y \longrightarrow x \star y$ **defines a complex structure on the orthogonal complement $x^\perp$.**

PROOF: This vector product is the octonion product on imaginary octonions. ■

CLAIM: For each non-zero vector $x \in V$, **its stabilizer $St_{G_2}(x) \cong SU(3)$**.

PROOF: $St_{G_2}(x)$ acts on $x^\perp$ preserving the complex structure and the metric, **hence $St_{G_2}(x) \subset U(3)$.** Its dimension is $\dim G_2 - \dim S^6 = 8$, hence $St_{G_2}(x) = SU(3) \subset U(3)$ (there are no other 8-dimensional subgroups in $U(3)$). ■

COROLLARY: For each non-zero vector $x \in V$, **the space $x^\perp$ is equipped with a natural $SU(3)$-structure.**

Complex Hermitian structure on $\operatorname{Knot}(M)$

COROLLARY: For each knot $S \subset M$ on a G_2-manifold, **its normal bundle NS is equiupped with an $SU(3)$-structure.**

PROOF: For each $x \in S$, $NS|_x = x^\perp$. ∎

REMARK: The space of sections $\Gamma(NS)$ of NS is, therefore, **a complex Hermitian vector space.**

THEOREM: Consider the space of knots on a G_2-manifold as a Frechet manifold, wih $T_S \operatorname{Knot}(M) = \Gamma(NS)$. Then **the Hermitian structure on $\Gamma(NS)$ defines a formally Kähler structure on $\operatorname{Knot}(M)$** if and only if M is a holonomy G_2-manifold.

REMARK: A symplectic structure was obtained by M. Movshev in 1999.

The Movshev symplectic structure

DEFINITION: Let $\mathsf{Knot}^m(M) \subset \mathsf{Knot}(M) \times M$ be the space of marked knots, that is, pairs $(S^1 \stackrel{\gamma}{\hookrightarrow} \mathsf{Knot}(M), s \in S^1)$, where $|\gamma'| = const$. Clearly, the forgetful map $\mathsf{Knot}^m(M) \stackrel{\pi}{\longrightarrow} \mathsf{Knot}(M)$ is an S^1-fibration. The fiberwise integration map

$$\Lambda^i(\mathsf{Knot}^m(M)) \stackrel{\pi_*}{\longrightarrow} \Lambda^{i-1}(\mathsf{Knot}(M))$$

is defined as usual,

$$\pi_*(\alpha)|_S := \int_{\pi^{-1}(S)} \left(\alpha \lrcorner \frac{d}{dt}\right) dt$$

where t is a parameter on S.

REMARK: The pushforward map π_* **always commutes with the de Rham differential.** This gives an interesting map

$$\pi_*\sigma^* : \Lambda^i(M) \longrightarrow \Lambda^{i-1}(\mathsf{Knot}(M))$$

commuting with the de Rham differential.

DEFINITION: Let (M, ρ) be a G_2-manifold. **The Movshev 2-form** on $\mathsf{Knot}(M)$ is defined as $\pi_*\sigma^*(\rho)$. It is closed iff $d\rho = 0$.

The Kähler form on Knot(M)

CLAIM: Let (M, ρ) be a G_2-manifold, $S \in$ Knot(M) a knot, and $\alpha, \beta \in NS$ two sections of a normal bundle, considered as tangent vectors $a, b \in T_S$ Knot(M). Consider the integral $S(a,b) := \int_S \rho(a,b,\cdot)|_S$. Then

$$\pi_* \sigma^*(\rho)(a,b) = S(a,b).$$

COROLLARY: **The Movshev symplectic form is equal to the Hermitian form on Knot(M).**

REMARK: Closedness of ρ implies the $d\omega = 0$ condition.

It remains to prove integrability of the complex structure on Knot(M) (equivalent to holonomy condition on M).

Non-degenerate (3,0)-forms

DEFINITION: Let M be a manifold (smooth or a Fréchet one) equipped with an almost complex structure, and $\Omega \in \Lambda^{3,0}(M)$ a $(3,0)$-form. We say that Ω is **non-degenerate** if for any $X \in T^{1,0}(M)$ there exist $Y, Z \in T^{1,0}(M)$ such that $\Omega(X,Y,Z) \neq 0$.

THEOREM: Let (M, I) be a manifold (smooth or a Fréchet one) equipped with an almost complex structure, and $\Omega \in \Lambda^{3,0}(M)$ a non-degenerate $(3,0)$-form. Assume that $d\Omega = 0$. **Then I is formally integrable.**

PROOF: Let $X, Y \in T^{1,0}M$ and $Z, T \in T^{0,1}(M)$. Since Ω is a (3,0)-form, it vanishes on $(0,1)$-vectors. Then Cartan's formula together with $d\Omega = 0$ implies that

$$0 = d\Omega(X,Y,Z,T) = \Omega(X,Y,[Z,T]).$$

From non-degeneracy of Ω we obtain that unless $[Z,T] \in T^{0,1}(M)$, for some $X, Y \in T^{1,0}M$ one would have $\Omega(X,Y,[Z,T]) \neq 0$. **Therefore, $[Z,T] \in T^{0,1}(M)$.** ∎

Constructing a non-degenerate (3,0)-form

CLAIM: Let $V = \mathbb{R}^7$ be a 7-dimensional real space equipped with a 3-form ρ, with $St_{GL(7)}(\rho) = G_2$, $\rho^* \in \Lambda^4\mathbb{R}^7$ the dual form, and $x \in V$ a unit vector. Consider $x^\perp$ equipped with a complex structure via the octonion vector product. **Then $\Omega := \rho + \sqrt{-1}\,\rho^* \lrcorner x$ is a holomorphic volume form on $x^\perp$.** Therefore, Ω is a non-degenerate (3,0)-form.

DEFINITION: Let M be a G_2-manifold, and Ω a 3-form defined as $\Omega := \xi + \sqrt{-1}\,\pi_*\sigma^*(\rho^*)$. Here $\xi(x,y,z)|_S := \int_S \rho(x,y,z)dt$, for each $S \in \operatorname{Knot}(M)$, where dt is a unit 1-form on S.

COROLLARY: For any G_2-manifold, Ω is a non-degenerate (3,0)-form.

PROOF: Follows from the above claim. ■

THEOREM: Ω is closed if and only if ρ and ρ^* are closed.

This theorem implies formal integrability of the complex structure on $\operatorname{Knot}(M)$.

APPLICATIONS

DEFINITION: Let M be a G_2-manifold, and $\Lambda^2 M = \Lambda^2_7(M) \oplus \Lambda^2_{14}(M)$ the irreducible decomposition of the bundle of 2-forms $\Lambda^2(M)$ associated with the G_2-action. A vector bundle (B, ∇) with connection is called **a G_2-instanton** if its curvature lies in $\Lambda^2_{14}(M) \otimes \operatorname{End}(B)$.

DEFINITION: Let (F, I) be a formally complex Fréchet manifold, and (B, ∇) a Hermitian bundle with connection. We say that (B, ∇) is **formally holomorphic** if the curvature Θ of ∇ satisfies $\Theta \in \Lambda^{1,1}(F) \otimes \operatorname{End} B$.

THEOREM: Let M be a G_2-manifold, $\operatorname{Knot}(M)$ its knot space equipped with a natural formally Kähler structure, (B, ∇) a Hermitian vector bundle with connection, and $(\tilde{B}, \tilde{\nabla})$ the corresponding bundle on $\operatorname{Knot}(M)$. **Then $(\tilde{B}, \tilde{\nabla})$ is formally holomorphic if and only if ∇ is a G_2-instanton.**

MORE APPLICATIONS

DEFINITION: Let $V = \mathbb{R}^7$ be a 7-space equipped with a 3-form ρ and the associated octonionic vector product. A 3-dimensional subspace $A \subset V$ is called **associative** if it is closed under the vector product. The set of associative subspaces is in bijective correspondence with the set of quaternionic subalgebras in octonions.

DEFINITION: Let $X \subset M$ be a 3-dimensional subvariety of a G_2-manifold. We say that X is **associative** if $T_xX \subset T_xM$ is an associative subspace for each smooth point $x \in X$.

PROPOSITION: Let M be a holonomy G_2-manifold, and $S \subset \operatorname{Knot}(M)$ a 1-dimensional complex subvariety. Denote by $\tilde{S} \subset M$ the union of all knots in S. **Then $\tilde{S}$ is an associative subvariety of M.**

PROPOSITION: Let M be a holonomy G_2-manifold, and $X \subset M$ a subvariety, $1 < \dim X < 7$. **Then $\operatorname{Knot}(X) \subset \operatorname{Knot}(M)$ is a formally complex subvariety if and only if X is an associative subvariety.**