Derived brackets and generalized complex manifolds

Misha Verbitsky

Abstract
This is a short note purporting to explain the generalized complex geometry through superalgebra and Clifford multiplication.

Contents

There is nothing essentially new in this note. It is a write-up of a talk given June 27 at "Geometric structures" seminar in Moscow, HSE. The idea was to explain the definition of Courant bracket in terms of spinors, and prove some basic results using this description. The basic reference is [G], [H] and [KS].

1 Clifford algebras

Definition 1.1: A Clifford algebra $\operatorname{Cl}(V, q)$ of a vector space V with a scalar product q is an algebra generated by V with a relation $xy + yx = q(x, y)1$.

Example 1.2: Suppose that $q = 0$. Then $xy = -yx$, hence the Clifford algebra $\operatorname{Cl}(V, q)$ is isomorphic to the Grassmann algebra: $\operatorname{Cl}(V, q) = \Lambda^* V$.

Example 1.3: Denote the k-dimensional space $\mathbb{R}^k$ with a scalar product of signature (q, p) by $(\mathbb{R}^n, \underbrace{+, ..., +}_{q}, \underbrace{-, ..., -}_{p})$. Clearly

$$\operatorname{Cl}(\mathbb{R}, -) = \mathbb{R}[t]/(t^2 = -1) = \mathbb{C},$$

and

$$\mathrm{Cl}(\mathbb{R}, +) = \mathbb{R}[t]/(t^2 = 1) = \mathbb{R} \oplus \mathbb{R}.$$

Exercise 1.4: Prove that $\mathrm{Cl}(\mathbb{R}^2, -, -)$ is isomorphic to the quaternion algebra, and $\mathrm{Cl}(\mathbb{R}^2, +, +)$, $\mathrm{Cl}(\mathbb{R}^2, +, -)$ are isomorphic to the algebra of 2x2-matrices, $\mathrm{Cl}(\mathbb{R}^2, +, +) \cong \mathrm{Cl}(\mathbb{R}^2, +, -) \cong \mathrm{Mat}(2, \mathbb{R})$.

Proposition 1.5: $\dim \mathrm{Cl}(V) = 2^{\dim V}$.

Before I give a proof of this result, let me introduce the filtered algebras.

Definition 1.6: Let $A_0 \subset A_1 \subset A_2 \subset ...$ be a sequence of subspaces of an algebra $A = \bigcup A_i$. We say that $\{A_i\}$ is **a multiplicative filtration** if $A_i \cdot A_j \subset A_{i+j}$. In this case A is called **a filtered algebra**.

Exercise 1.7: Prove that the direct sum $\bigoplus_i A_i/A_{i-1}$ is equipped with an algebra structure: $a \in A_i \mod A_{i-1}$ multiplied by $a' \in A_j \mod A_{j-1}$ gives $aa' \in A_{ij} \mod A_{ij-1}$.

Definition 1.8: Let $A_0 \subset A_1 \subset A_2 \subset ... \subset A$ be a filtered algebra. **Associated graded algebra** of this filtration is $\bigoplus_i A_i/A_{i-1}$ with the algebra structure defined above.

Claim 1.9: Let $\mathrm{Cl}(V, q)$ be a Clifford algebra, and $\mathrm{Cl}_0(V, q) = k \cdot 1$ the field of constants, $\mathrm{Cl}_1(V) = \mathrm{Cl}_0(V, q) \oplus V$, and $\mathrm{Cl}_i(V, q) := \underbrace{\mathrm{Cl}_1(V, q), ..., Cl_1(V, q)}_{i \text{ times}}$. This gives a filtration on $\mathrm{Cl}(V, q)$. Then the associated graded algebra is the Grassmann algebra $\Lambda^* V$.

Proof: Modulo lower terms of the filtration, the Clifford relations give $xy + yx = 0$. ■

Now the proof of Proposition 1.5 is apparent; indeed, taking associated graded algebra does not change the dimension, hence $\dim \mathrm{Cl}(V, q) = \dim \Lambda^* V = 2^{\dim V}$.

Theorem 1.10: Let $V := W \oplus W^*$, with the usual pairing $\langle (x + \xi), (x' + \xi') \rangle = \xi(x') + \xi'(x)$. Then $\mathrm{Cl}(V)$ is naturally isomorphic to $\mathrm{Mat}(\Lambda^* V^*)$.

Proof: Consider the convolution map $W \otimes \Lambda^i W^* \longrightarrow \Lambda^{i-1} W^*$, with $v \otimes \xi \longrightarrow \xi(v, \cdot, \cdot, \ldots, \cdot)$ denoted by $v, \xi \longrightarrow i_v(\xi)$ and the extertior multiplication map $W^* \otimes \Lambda^i W^* \longrightarrow \Lambda^{i-1} W^*$, with $\nu \otimes \xi \longrightarrow \nu \wedge \xi$, denoted by $\nu, \xi \longrightarrow e_\nu(\xi)$. Let $V \otimes \Lambda^* W^* \xrightarrow{\Gamma} \Lambda^* W^*$ map $(v, \nu) \otimes \xi$ to $i_v(\xi) + e_\nu(\xi)$. It is easy to check that all i_v pairwise anticommute, all e_ν pairwise anticommute, and the anticommutator $\{i_v, e_\nu\}$ is a scalar operator of multiplication by a number $\nu(v)$.

To prove the last assertion without any calculations, we notice that i_v is an odd derivation of the Grassmann algebra, e_ν is a linear operator, and a commutator of a derivation and a linear operator is linear, hence one has

$$\{i_v, e_\nu\}(a) = \{i_v, e_\nu\}(1) \wedge a = \nu(v) \cdot a.$$

These anticommutator relations immediately imply that the map $V \otimes \Lambda^* W^* \xrightarrow{\Gamma} \Lambda^* W^*$, called **the Clifford multiplication map**, is extended to a homomorphism $\mathrm{Cl}(V) \longrightarrow \mathrm{Mat}(\Lambda^* W^*)$. An elementary calculation (left as an exercise) proves that this map is injective. Since $\dim \mathrm{Cl}(V) = 2^{\dim V} = 2^{2\dim W} = \dim \mathrm{Mat}(\Lambda^* W^*)$, this also implies that $\mathrm{Cl}(V) \cong \mathrm{Mat}(\Lambda^* W^*)$. ■

Definition 1.11: Let (V, q) be a vector space equipped with a scalar product, and $\mathrm{Cl}(V, q) \cong \mathrm{Mat}(S)$ an isomorphism (such an isomorphism is possible only when V is even-dimensional, because otherwise $2^{\dim V}$ is not a square of anything). Then S is called **the space of spinors over** (V, q).

Remark 1.12: The Lie group $SO(V, q)$ acts on $\mathrm{Cl}(V, q)$ by automorphisms. However, $\mathrm{Aut}(\mathrm{Mat}(S)) = PSL(S)$ (this is left as an exercise). This gives a group homomorphism $SO(V, q) \longrightarrow PSL(S)$. Lifting this homomorphism to the universal covering $\mathrm{Spin}(V) \longrightarrow SL(S)$, we obtain **the spinorial representation** of the spin group $\mathrm{Spin}(V)$; it is a smallest faithfull representation of the spin group.

Definition 1.13: Let M be a smooth manifold. Consider the Clifford multiplication $V \otimes S \longrightarrow S$. Apply this construction to the bundle $\Lambda^* M$ taken, fiberwise, as spinors over $V := TM \oplus T^*M$. We obtain the Clifford multiplication map $\Gamma : V \otimes \Lambda^* M \longrightarrow \Lambda^* M$ written as $(v, \nu) \otimes \xi \stackrel{\Gamma}{\mapsto} i_v(\xi) + e_\nu(\xi)$.

2 Derived brackets

2.1 Graded algebras

Definition 2.1: A **graded vector space** is a space $V^* = \bigoplus_{i \in \mathbb{Z}} V^i$.

Remark 2.2: If V^* is graded, the endomorphisms space

$$\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$$

is also graded, with

$$\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j}).$$

Definition 2.3: A **graded algebra** (or "graded associative algebra") is an associative algebra $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$, with the product compatible with the grading: $A^i \cdot A^j \subset A^{i+j}$.

Definition 2.4: A bilinear map of graded spaces which satisfies $A^i \cdot A^j \subset A^{i+j}$ is called **graded**, or **compatible with grading**.

Remark 2.5: The category of graded spaces can be defined as a category of vector spaces with $U(1)$-action, with the weight decomposition providing the grading. Then a graded algebra is an associative algebra in the category of spaces with $U(1)$-action.

Definition 2.6: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** $\tilde{a}$ of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

Definition 2.7: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a, b\} = ab - (-1)^{\tilde{a}\tilde{b}} ba$.

Definition 2.8: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

Example 2.9: The Grassmann algebra is supercommutative.

Definition 2.10: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \ \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a, b\} = -(-1)^{\tilde{a}\tilde{b}}\{b, a\}$ and satisfies **the super Jacobi identity** $\{c, \{a, b\}\} = \{\{c, a\}, b\} + (-1)^{\tilde{a}\tilde{c}}\{a, \{c, b\}\}$

Example 2.11: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. Then $\operatorname{End}(A^*), \{\cdot,\cdot\}$ is a graded Lie algebra.

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d, d\} = 0$, and L an even or odd element. Then $\{\{L, d\}, d\} = 0$.

Proof: $0 = \{L, \{d, d\}\} = \{\{L, d\}, d\} + (-1)^{\tilde{L}}\{d, \{L, d\}\} = 2\{\{L, d\}, d\}$. ∎

2.2 Loday bracket

For the history and different versions of the definition of derived brackets please see [KS]. For the present purposes, we need only one of them, namely the Loday bracket.

From now on, we use the notation $[\cdot, \cdot]$ for the supercommutator.

Definition 2.12: Let $A = \bigoplus A^i$ be a Lie superalgebra, and $d: A \longrightarrow A$ an odd endomorphism satisfying $d^2 = 0$. Define **the Loday bracket** $[a, b]_d := (-1)^{\tilde{a}}[d(a), b]$.

Exercise 2.13: Prove that the Loday bracket satisfies the graded Jacobi identity:

$$[a, [b, c]_d]_d = [[a, b]_d, c]_d + (-1)^{\tilde{a}\tilde{b}}[b, [a, c]_d]_d.$$

Example 2.14: Now let A be the superalgebra $\operatorname{End}(\Lambda^* M)$, where M is a smooth manifold, and d the de Rham differential. Cartan formulas can be written in terms of Loday bracket as follows (we use the notation e_ν, i_v introduced in Definition 1.13).

$$\begin{aligned}[]
[i_x, i_y]_d &= i_{[x,y]} \\
[e_\eta, i_x]_d &= [e_{d\eta}, i_x] = e_{d\eta \lrcorner x} \\
[i_x, e_\eta]_d = \operatorname{Lie}_x e_\eta &= e_{\operatorname{Lie}_x \eta}
\end{aligned}$$

for all $x, y \in TM, \eta \in \Lambda^1 M$.

Proof: Cartan's formula gives $[d, i_v] = \operatorname{Lie}_v$ (note that $[\cdot, \cdot]$ here denotes a supercommutator). Then $[i_x, i_y]_d = \operatorname{Lie}_x i_y = i_{\operatorname{Lie}_x y} = i_{[x,y]}$.

Since i_x is a derivation of the de Rham algebra (prove this), the commutator $[i_x, e_\eta]$ is linear, and this gives $[i_x, e_\xi](a) = i_x e_\xi(1) \cdot a = \xi \lrcorner x$ where $\xi \lrcorner x$ is contraction, $\xi \lrcorner x = i_x(\xi)$. This takes care about the formula $[e_\eta, i_x]_d = e_{d\eta \lrcorner x}$.

Finally, the last formula is self-evident. ∎

Corollary 2.15: Let $V := TM \oplus T^*M$. Consider the Clifford multiplication map $\Gamma : V \otimes \Lambda^* M \longrightarrow \Lambda^* M$, and let $x, x' \in V$, with $x = (x, \nu), x' = (v', \nu')$. Then $[\Gamma_x, \Gamma_{x'}]_d = \Gamma_y$, where $y = ([v, v'], (d\nu) \lrcorner v - \operatorname{Lie}_{v'} \nu)$. ■

Definition 2.16: We define **the Courant bracket** $[(v,\nu),(v',\nu')]_d := ([v,v'], (d\nu) \lrcorner v - \operatorname{Lie}_{v'} \nu)$

Remark 2.17: From Corollary 2.15 it is apparent that the Courant bracket is the Loday bracket applied to the Clifford multiplication operators.

Exercise 2.18: Prove that $[u, v]_d - [v, u]_d = d\langle u, v\rangle$.

Remark 2.19: The skew-symmetric bracket $[u, v]_d - [u, v]_d$ is called **the Dorfman bracket**, after I. Ya. Dorfman.

2.3 Complex structures and generalized complex structures

Definition 2.20: Let V be a real vector space. **A complex structure operator** on V is $I \in \operatorname{Hom}(V, V)$ satisfying $I^2 = -\operatorname{Id}_V$.

Claim 2.21: The eigenvalues α_i of I are $\pm\sqrt{-1}$. Moreover, I diagonalizable over $\mathbb{C}$.

Proof: The operator I is unitary with respect to the Hermitian form $g_I(x, y) := g(x, y) + g(Ix, Iy)$, where g is an arbitrary Hermitian form. Any unitary matrix is diagonalizable. Finally, $\alpha_i^2 = -1$, because $I^2 = -\operatorname{Id}$. ■

Definition 2.22: Let V be a vector space, and $I \in \operatorname{End}(V)$ a complex structure operator. The eigenvalue decomposition $V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$ is called **the Hodge decomposition**; here $I\Big|_{V^{1,0}} = \sqrt{-1}\operatorname{Id}$, and $I\Big|_{V^{0,1}} = -\sqrt{-1}\operatorname{Id}$.

Remark 2.23: One can reconstruct I from the space $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$. Indeed, take $V^{0,1} = \overline{V^{1,0}}$, and let I act on $V^{0,1}$ as $\sqrt{-1}\operatorname{Id}$, and on $V^{0,1}$ as $-\sqrt{-1}\operatorname{Id}$. Since thus defined operator $I \in \operatorname{End}(V \otimes_{\mathbb{R}} \mathbb{C})$ commutes with the complex conjugation, it is **real**, that is, preserves $V \subset V \otimes_{\mathbb{R}} \mathbb{C}$. This gives an identification between the set of complex structures on $V, \dim_{\mathbb{R}} V = 2n$, and an open part of the Grassmann space $Gr_n(V \otimes_{\mathbb{R}} \mathbb{C})$ consisting of all subspaces $W \subset V \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $W \cap \overline{W} = 0$.

Definition 2.24: **An almost complex structure** on a real $2n$-manifold M is an operator $I \in \operatorname{End}(TM)$ satisfying $I^2 = -\operatorname{Id}_{TM}$, or, equivalently, an

n-dimensional sub-bundle $T^{1,0}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ such that $T^{1,0}M \cap \overline{T^{1,0}M} = 0$. The almost complex structure called **integrable** (and M **a complex manifold**) if $T^{1,0}M$ is *involutive*, that is, satisfies $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

Definition 2.25: Let M be a real $2n$-manifold, and $V = TM \oplus T^*M$. Consider the standard symmetric pairing on V of signature $(2n, 2n)$,

$$\langle (v, \nu), (v', \nu') \rangle := \nu(v') + \nu'(v).$$

Let $I \in \operatorname{End} V$ an orthogonal operator satisfying $I^2 = -\operatorname{Id}_V$. Then I is called **a generalized almost complex structure**.

Definition 2.26: Let V be an even-dimensional vector space equipped with a non-degenerate scalar product h, and $W \subset V$ a subspace. Then W is called **isotropic** if $h\Big|_W = 0$, and **maximal isotropic** if $\dim W = \frac{1}{2} \dim V$.

Exercise 2.27: Prove the dimension of an isotropic subspace is always $\leqslant \frac{1}{2} \dim V$.

Remark 2.28: Let $V = TM \oplus T^*M$, $I \in \operatorname{End}(V)$ a generalized almost complex structure, and $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ be the $\sqrt{-1}$-eigenspace. Then $V^{1,0}$ is maximal isotropic. Indeed, $\langle v, v' \rangle = \langle Iv, Iv' \rangle = -\langle v, v' \rangle$ for all $v, v' \in V^{1,0}$.

Claim 2.29: Let M be a smooth manifold, $V = TM \oplus T^*M$. The generalized almost complex structures $I \in \operatorname{End}(V)$ are in bijective corresponidence with maximal isotropic subbundles $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $V^{1,0} \cap \overline{V^{1,0}} = 0$.

Proof: $I \in \operatorname{End}(V)$ a generalized almost complex structure, and $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ its $\sqrt{-1}$-eigenspace. As shown in Remark 2.28, $V^{1,0}$ is maximal isotropic. It remains to show that this correspondence is bijective.

Let $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ be a maximal isotropic subbundle, satisfying $V^{1,0} \cap \overline{V^{1,0}} = 0$. Then $V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus \overline{V^{1,0}}$. Define $I \in \operatorname{End}(V)$ using $I\Big|_{V^{1,0}} = \sqrt{-1}\operatorname{Id}$, and $I\Big|_{\overline{V^{1,0}}} = -\sqrt{-1}\operatorname{Id}$. Then $I^2 = -\operatorname{Id}_V$; to prove Claim 2.29 it remains only to show that I is orthogonal.

However, $\langle \cdot, \cdot \rangle\Big|_{V^{1,0}} = \langle \cdot, \cdot \rangle\Big|_{\overline{V^{1,0}}} = 0$, because $V^{1,0}$ is isotropic, and for any $v \in V^{1,0}$, $v' \in \overline{V^{1,0}}$, one has $\langle Iv, Iv' \rangle = \langle \sqrt{-1}\, v, -\sqrt{-1}\, v' \rangle = \langle v, v' \rangle$. ■

Definition 2.30: A generalized almost complex structure I on M is **integrable** if $[V^{1,0}, V^{1,0}]_d \subset V^{1,0}$. Then I is called **a generalized complex structure**, and M **a generalized complex manifold**.

The following examples explain the utility of generalized complex structures, which unite into one usable definition the notions of complex and symplectic structures. Their integrability remains to be proven after the pure spinors are introduced and used to prove integrability.

Example 2.31: Let (M,ω) be a symplectic manifold. Consider an almost complex structure $I \in \operatorname{End}(TM \oplus T^*M)$ written as

$$I := \begin{pmatrix} 0 & -\omega \\ \omega^{-1} & 0 \end{pmatrix}.$$

Then I is integrable.

Remark 2.32: Integrability of the generalized almost complex structure is in fact equivalent to $d\omega = 0$, which is an easy exercise using the pure spinor approach.

Example 2.33: Let (M,J) be a complex manifold. Consider an almost complex structure $I \in \operatorname{End}(TM \oplus T^*M)$ written as

$$I := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}.$$

Then I is integrable.

3 Pure spinors and generalized complex structures

Remark 3.1: Let (V,h) be a vector space equipped with a scalar product, S the corresponding spinors, and $V \otimes S \longrightarrow S$ the Clifford multiplication. Given a non-zero spinor $\Psi \in S$, consider the space

$$\ker\Psi : \ \{v \in V \ \ | \ \ v \cdot Psi = 0\}.$$

Then $\ker\Psi$ is isotropic. Indeed, for each $u, v \in \ker\Psi$, one has $0 = uv\Psi + uv\Psi = h(u,v)\Psi$.

Definition 3.2: (Cartan, Chevalley) $\Psi \in S$ is **a pure spinor** if $\ker\Psi$ is maximal isotropic.

Remark 3.3: The following theorem gives a Plücker-type embedding for the maximally isotropic Grassmannian.

Theorem 3.4: (Chevalley) Let (V,h) be a vector space equipped with a scalar product, and S its spinor space. Then for each maximally isotropic subspace $W \subset V$, $W = \ker \Psi$ for some pure spinor $\Psi \in S$, which is unique up to a scalar multiplier.

Proof: Identifying V with $W \oplus W^*$, we obtain an identification $S = \Lambda^* W$ (Theorem 1.10). Let $w_1, ..., w_n$ be a basis in W. Then $\ker w_1 \wedge w_2 \wedge ... \wedge w_n = W$.

Converse is also obvious: if $\Psi \in \Lambda^* W$ satisfies $W \wedge \Psi = 0$, one has $\Psi = C w_1 \wedge w_2 \wedge ... \wedge w_n$. ■

Example 3.5: Let (M, J) be a complex n-manifold, and I the generalized complex structure constructed as in Example 2.33. The corresponding pure spinor is any non-degenerate section of $\Lambda^{n,0}(M, J)$ (check this).

Example 3.6: Let (M, ω) be a symplectic n-manifold, and I the generalized complex structure constructed as in Example 2.31. The corresponding pure spinor is $\Psi = e^{\sqrt{-1}\,\omega}$. Indeed, $V^{1,0}$ is spanned by $i_x - \sqrt{-1}\, e_{\omega \lrcorner x}$, where $x \in TM$. Since i_x is a derivation, one has $i_x(e^{\sqrt{-1}\,\omega}) = \sqrt{-1}\, i_x(\omega) \wedge e^{\sqrt{-1}\,\omega}$, giving

$$i_x - \sqrt{-1}\, e_{\omega \lrcorner x}(e^{\sqrt{-1}\,\omega}) = \ 1 i_x(\omega) e^{\sqrt{-1}\,\omega} - \sqrt{-1}\, e_{\omega \lrcorner x}(e^{\sqrt{-1}\,\omega}) = 0.$$

Theorem 3.7: Let L be a maximal isotropic subbundle in $V = TM \oplus T^*M$, and $\Psi \in \Lambda^* M$ the corresponding spinor. Then L satisfies $[L, L]_d \subset L$ if and only if $d\Psi = t \cdot \Psi$, for some $t \in V$.

Proof: $u, v \in \ker \Psi$, then

$$[u, v]_d \Psi = [du + ud, v]\Psi = duv\Psi + udv\Psi - vdu\Psi - vud\Psi = -vud\Psi.$$

If $d\Psi = 0$, one has $[u, v]_d \Psi = -vud\Psi = 0$.

To prove the converse, consider the filtration on the spinor bundle, $S_0 = \Psi, S_1 = V \cdot \Psi, ..., S_d = V \cdot S_{d-1}$. Denote $\ker \Psi$ by $V^{1,0}$. Let $\Lambda^d V^{1,0} \subset \mathrm{Cl}(V)$ be the subspace in the Clifford algebra generated by the monomials of degree d on $V^{1,0}$. Clearly, $S_d = \{s \in S \ \mid\ \Lambda^d V^{1,0} s = 0\}$.

Let now $v, u, [u, v]_d \in \ker \Psi$. The same calculation as above gives $-vud\Psi = 0$. This implies that $d\Psi \in S_1$ for all pure spinors Ψ inducing integrable generalized complex structure. However, $S_1 = V \cdot \Psi$. ■

References

[G] Marco Gualtieri, *Generalized complex geometry*, arXiv:math/0401221

[H] Nigel Hitchin *Generalized Calabi-Yau manifolds*, arXiv:math/0209099, Quart. J. Math. Oxford Ser. 54:281-308, 2003.

[KS] Yvette Kosmann-Schwarzbach, *Derived brackets*, arXiv:math/0312524.

Misha Verbitsky
Laboratory of Algebraic Geometry,
National Research University HSE,
Faculty of Mathematics, 7 Vavilova Str. Moscow, Russia,
`verbit@mccme.ru`, also:
Kavli IPMU (WPI), the University of Tokyo