Generalized complex structures and derived brackets

Misha Verbitsky

Geometric structures on manifolds,

IMPA, 11.04.2024

Clifford algebras

DEFINITION: A Clifford algebra $\mathcal{C}l(V,q)$ of a vector space V with a scalar product q is an algebra generated by V with a relation $xy+yx=q(x,y)1$.

EXAMPLE: Suppose that $q=0$. Then $xy=-yx$, hence **the Clifford algebra $\mathcal{C}l(V,q)$ for $q=0$ is isomorphic to the Grassmann algebra: $\mathcal{C}l(V,q)=\Lambda^* V$.**

EXAMPLE: Denote the k-dimensional space $\mathbb{R}^k$ with a scalar product of signature (q,p) by $(\mathbb{R}^n, \underbrace{+,...,+}_q, \underbrace{-,...,-}_p)$. **Clearly,** $\mathcal{C}l(\mathbb{R},-)=\mathbb{R}[t]/(t^2=-1)=\mathbb{C}$, **and** $\mathcal{C}l(\mathbb{R},+)=\mathbb{R}[t]/(t^2=1)=\mathbb{R}\oplus\mathbb{R}$.

EXERCISE: Prove that $\mathcal{C}l(\mathbb{R}^2,-,-)$ is isomorphic to the quaternion algebra, and $\mathcal{C}l(\mathbb{R}^2,+,+)$, $\mathcal{C}l(\mathbb{R}^2,+,-)$ are isomorphic to the algebra of 2x2-matrices, $\mathcal{C}l(\mathbb{R}^2,+,+)\cong\mathcal{C}l(\mathbb{R}^2,+,-)\cong \mathrm{Mat}(2,\mathbb{R})$.

Filtered algebras

DEFINITION: Let $A_0 \subset A_1 \subset A_2 \subset ...$ be a sequence of subspaces of an algebra $A = \bigcup A_i$. We say that $\{A_i\}$ is **a multiplicative filtration** if $A_i \cdot A_j \subset A_{i+j}$. In this case A is called **a filtered algebra**.

EXERCISE: Prove that **the direct sum $\bigoplus_i A_i/A_{i-1}$ is equipped with an algebra structure:** $a \in A_i \mod A_{i-1}$ multiplied by $a' \in A_j \mod A_{j-1}$ gives $aa' \in A_{ij} \mod A_{ij-1}$.

DEFINITION: Let $A_0 \subset A_1 \subset A_2 \subset ... \subset A$ be a filtered algebra. **Associated graded algebra** of this filtration is $\bigoplus_i A_i/A_{i-1}$ with the algebra structure defined above.

CLAIM: Let $\mathcal{C}l(V,q)$ be a Clifford algebra, and $\mathcal{C}l_0(V,q) = k \cdot 1$ the field of constants, $\mathcal{C}l_1(V) = \mathcal{C}l_0(V,q) \oplus V$, and $\mathcal{C}l_i(V,q) := \underbrace{\mathcal{C}l_1(V,q), ..., Cl_1(V,q)}_{i \text{ times}}$. This gives a filtration on $\mathcal{C}l(V,q)$. **Then the associated graded algebra is the Grassmann algebra $\Lambda^* V$.**

Proof: Modulo lower terms of the filtration, the Clifford relations give $xy + yx = 0$. ■

COROLLARY: $\dim \mathcal{C}l(V) = 2^{\dim V}$.

$\mathcal{C}l(W \oplus W^*)$

THEOREM: Let $V := W \oplus W^*$, with the usual pairing $\langle (x+\xi), (x'+\xi')\rangle = \xi(x') + \xi'(x)$. **Then $\mathcal{C}l(V)$ is naturally isomorphic to $\operatorname{Mat}(\Lambda^* V^*)$.**

Proof. Step 1: Consider the convolution map $W \otimes \Lambda^i W^* \longrightarrow \Lambda^{i-1} W^*$, with $v \otimes \xi \longrightarrow \xi(v, \cdot, \cdot, \ldots, \cdot)$ denoted by $v, \xi \longrightarrow i_v(\xi)$ and the extertior multiplication map $W^* \otimes \Lambda^i W^* \longrightarrow \Lambda^{i-1} W^*$, with $\nu \otimes \xi \longrightarrow \nu \wedge \xi$, denoted by $\nu, \xi \longrightarrow e_\nu(\xi)$. Let $V \otimes \Lambda^* W^* \stackrel{\Gamma}{\longrightarrow} \Lambda^* W^*$ map $(v, \nu) \otimes \xi$ to $i_v(\xi) + e_\nu(\xi)$.

Then all i_v pairwise anticommute, all e_ν pairwise anticommute, and the anticommutator $\{i_v, e_\nu\}$ is a scalar operator of multiplication by a number $\nu(v)$.

To prove the last assertion without any calculations, we notice that **i_v is an odd derivation of the Grassmann algebra, e_ν is a linear operator, and a commutator of a derivation and a linear operator is linear,** hence one has

$$\{i_v, e_\nu\}(a) = \{i_v, e_\nu\}(1) \wedge a = \nu(v) \cdot a.$$

Step 2: These relation imply that the map $V \otimes \Lambda^* W^* \stackrel{\Gamma}{\longrightarrow} \Lambda^* W^*$, called **the Clifford multiplication map**, is extended to a homomorphism $\mathcal{C}l(V) \longrightarrow \operatorname{Mat}(\Lambda^* W^*)$. I's not hard to show that this map is surjective. **Since $\dim \mathcal{C}l(V) = 2^{\dim V} = 2^{2\dim W} = \dim \operatorname{Mat}(\Lambda^* W^*)$, this also implies that $\mathcal{C}l(V) \cong \operatorname{Mat}(\Lambda^* W^*)$.** ■

Spinoprial representation

REMARK: The Lie group $SO(V,q)$ acts on $\mathcal{C}l(V,q)$ by automorphisms. However, $\operatorname{Aut}(\operatorname{Mat}(S)) = PSL(S)$ (this is left as an exercise). This gives a group homomorphism $SO(V,q) \longrightarrow PSL(S)$. Lifting this homomorphism to the universal covering $\operatorname{Spin}(V) \longrightarrow SL(S)$, we obtain **the spinorial representation** of the spin group $\operatorname{Spin}(V)$; it is a smallest faithfull representation of the spin group.

DEFINITION: Let M be a smooth manifold, $V = TM \oplus T^*M$ and $S = \Lambda^*(M)$. Consider the Clifford multiplication $V \otimes S \longrightarrow S$, we obtain **the Clifford multiplication map** $\Gamma: V \otimes \Lambda^* M \longrightarrow \Lambda^* M$ written as $(v,\nu) \otimes \xi \stackrel{\Gamma}{\mapsto} i_v(\xi) + e_\nu(\xi)$.

Graded vector spaces

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$. If V^* is graded, **the endomorphisms space $\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$ is also graded,** with $\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j})$.

DEFINITION: A **graded algebra** (or "graded associative algebra") is an associative algebra $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$, with the product compatible with the grading: $A^i \cdot A^j \subset A^{i+j}$.

DEFINITION: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** $\tilde{a}$ of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a, b\} = ab - (-1)^{\tilde{a}\tilde{b}} ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

Loday bracket

From now on, **we use the notation $[\cdot,\cdot]$ for the supercommutator.**

DEFINITION: Let $A = \bigoplus A^i$ be a graded associative algebra, and $d: A \longrightarrow A$ an odd endomorphism satisfying $d^2 = 0$. Define **the Loday bracket** $[a,b]_d := (-1)^{\tilde{a}}[d(a),b]$.

EXERCISE: Prove that the **Loday bracket satisfies the graded Jacobi identity:**

$$[a,[b,c]_d]_d = [[a,b]_d,c]_d + (-1)^{\tilde{a}\tilde{b}}[b,[a,c]_d]_d.$$

Loday gracket on endomorphisms of de Rham algebra

CLAIM: Now let A be the graded algebra End$(\Lambda^* M)$, where M is a smooth manifold, and d the de Rham differential, acting on A as $d(a) = [d, a]$. **Then** $[e_\eta, e_{\eta'}]_d = 0$ **and**

$$\begin{aligned}
[i_x, i_y]_d =& i_{[x,y]} \\
[e_\eta, i_x]_d =& [e_{d\eta}, i_x] = e_{i_x(d\eta)} \\
[i_x, e_\eta]_d =& \text{Lie}_x\, e_\eta = e_{\text{Lie}_x \eta}
\end{aligned}$$

for all $x, y \in TM, \eta, \eta' \in \Lambda^1 M$**.**

Proof. Step 1: Cartan's formula gives $[d, i_v] = \text{Lie}_v$ (we use $[\cdot,\cdot]$ for the supercommutator). Then $[i_x, i_y]_d = [\text{Lie}_x, i_y] = i_{\text{Lie}_x y} = i_{[x,y]}$.

Step 2: Since i_x is a derivation of the de Rham algebra (prove this), the commutator $[i_x, e_\eta]$ is linear, and this gives $[i_x, e_\xi](a) = i_x e_\xi(1) \cdot a = i_x(\xi) \cdot a$. Then $[e_\eta, i_x]_d = e_{d\eta \lrcorner x}$.

Step 3: The last formula follows from $[d, i_x] = \text{Lie}_x$ and $[\text{Lie}_x, e_\eta] = e_{\text{Lie}_x \eta}$. ∎

Courant bracket

COROLLARY: Let $V := TM \oplus T^*M$. Consider the Clifford multiplication map $\Gamma: V \otimes \Lambda^*M \longrightarrow \Lambda^*M$, and let $x, x' \in V$, with $x = (x, \nu), x' = (v', \nu')$. **Then** $[\Gamma_x, \Gamma_{x'}]_d = \Gamma_y$, **where** $y = ([v, v'], i_v(d\nu') - \operatorname{Lie}_{v'} \nu)$.

DEFINITION: We define **the Courant bracket** on $TM \oplus T^*M$:

$$[(v, \nu), (v', \nu')]_d := ([v, v'], i_v(d\nu') - \operatorname{Lie}_{v'} \nu).$$

REMARK: The **Courant bracket is Loday bracket** applied to the Clifford multiplication operators.

CLAIM: $[a, b]_d + [v, u]_d = -d\langle a, b\rangle$

Proof: $d\langle x, \eta\rangle = d(i_x \eta) = \operatorname{Lie}_x \eta - i_x(d\eta)$ ■

REMARK: The skew-symmetric bracket $[a, b]_D := [a, b]_d - [b, a]_d$ is called **the Dorfman bracket**, after I. Ya. Dorfman.

Complex structures

DEFINITION: Let V be a real vector space. **A complex structure operator** on V is $I \in \operatorname{Hom}(V,V)$ satisfying $I^2 = -\operatorname{Id}_V$.
CLAIM: The eigenvalues α_i of I are $\pm\sqrt{-1}$. Moreover, I **diagonalizable over $\mathbb{C}$.** ■

DEFINITION: Let V be a vector space, and $I \in \operatorname{End}(V)$ a complex structure operator. The eigenvalue decomposition $V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$ is called **the Hodge decomposition**; here $I\big|_{V^{1,0}} = \sqrt{-1}\operatorname{Id}$, and $I\big|_{V^{0,1}} = -\sqrt{-1}\operatorname{Id}$.

REMARK: One can reconstruct I from the space $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$. Indeed, take $V^{0,1} = \overline{V^{1,0}}$, and let I act on $V^{0,1}$ as $\sqrt{-1}\operatorname{Id}$, and on $V^{0,1}$ as $-\sqrt{-1}\operatorname{Id}$. Since thus defined operator $I \in \operatorname{End}(V \otimes_{\mathbb{R}} \mathbb{C})$ commutes with the complex conjugation, it is **real**, that is, preserves $V \subset V \otimes_{\mathbb{R}} \mathbb{C}$. This gives **an identification between the set of complex structures on $V, \dim_{\mathbb{R}} V = 2n$, and an open part of the Grassmann space $Gr_n(V \otimes_{\mathbb{R}} \mathbb{C})$ consisting of all subspaces $W \subset V \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $W \cap \overline{W} = 0$.**

DEFINITION: An almost complex structure on a real $2n$-manifold M is an operator $I \in \operatorname{End}(TM)$ satisfying $I^2 = -\operatorname{Id}_{TM}$, or, equivalently, an n-dimensional sub-bundle $T^{1,0}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ such that $T^{1,0}M \cap \overline{T^{1,0}M} = 0$. The almost complex structure called **integrable** (and M **a complex manifold**) if $T^{1,0}M$ satisfies $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

Generalized almost complex structures

DEFINITION: Let M be a real $2n$-manifold, and $V = TM \oplus T^*M$. Consider the standard symmetric pairing on V of signature $(2n, 2n)$,

$$\langle (v,\nu), (v',\nu')\rangle := \nu(v') + \nu'(v).$$

Let $I \in \operatorname{End} V$ an orthogonal operator satisfying $I^2 = -\operatorname{Id}_V$. Then I is called **a generalized almost complex structure**.

DEFINITION: Let V be an even-dimensional vector space equipped with a non-degenerate scalar product h, and $W \subset V$ a subspace. Then W is called **isotropic** if $h|_W = 0$, and **maximal isotropic** if $\dim W = \frac{1}{2}\dim V$.

EXERCISE: Prove the **dimension of an isotropic subspace is always $\leqslant \frac{1}{2}\dim V$.**

REMARK: Let $V = TM \oplus T^*M$, $I \in \operatorname{End}(V)$ a generalized almost complex structure, and $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ be the $\sqrt{-1}$-eigenspace. **Then $V^{1,0}$ is maximal isotropic.** Indeed, $\langle v, v'\rangle = \langle Iv, Iv'\rangle = -\langle v, v'\rangle$ for all $v, v' \in V^{1,0}$.

Maximal isotropic subspaces

CLAIM: Let M be a smooth manifold, $V = TM \oplus T^*M$. The **generalized almost complex structures $I \in \operatorname{End}(V)$ are in bijective corresponidence with maximal isotropic subbundles** $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $V^{1,0} \cap \overline{V^{1,0}} = 0$.

Proof. Step 1: Let $I \in \operatorname{End}(V)$ a generalized almost complex structure, and $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ its $\sqrt{-1}$-eigenspace. As shown above, **$V^{1,0}$ is maximal isotropic.** It remains to show that this correspondence is bijective.

Step 2: Let $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ be a maximal isotropic subbundle, satisfying $V^{1,0} \cap \overline{V^{1,0}} = 0$. Then $V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus \overline{V^{1,0}}$. Define $I \in \operatorname{End}(V)$ using $I\big|_{V^{1,0}} = \sqrt{-1}\operatorname{Id}$, and $I\big|_{\overline{V^{1,0}}} = -\sqrt{-1}\operatorname{Id}$. Then $I^2 = -\operatorname{Id}_V$. **To prove that I is generalized almost complex, it remains only to show that I is orthogonal.**

Step 3: $\langle\cdot,\cdot\rangle\big|_{V^{1,0}} = \langle\cdot,\cdot\rangle\big|_{\overline{V^{1,0}}} = 0$, because $V^{1,0}$ is isotropic, and for any $v \in V^{1,0}$, $v' \in \overline{V^{1,0}}$, one has $\langle Iv, Iv'\rangle = \langle\sqrt{-1}\, v, -\sqrt{-1}\, v'\rangle = \langle v, v'\rangle$. ■

Generalized complex structures

DEFINITION: A generalized almost complex structure I on M is **integrable** if $[V^{1,0}, V^{1,0}]_d \subset V^{1,0}$. Then I is called **a generalized complex structure**, and M **a generalized complex manifold**.

CLAIM: Let ω be a non-degenerate 2-form on M. Consider an almost complex structure $I \in \operatorname{End}(TM \oplus T^*M)$ written as $I := \begin{pmatrix} 0 & -\omega \\ \omega^{-1} & 0 \end{pmatrix}$.

Then I is integrable if and only if $d\omega = 0$.

Proof: Later today (uses spinors). ■

CLAIM: Let (M, J) be a complex manifold. Consider an almost complex structure $I \in \operatorname{End}(TM \oplus T^*M)$ written as $I := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$. **Then I is integrable.**

Proof: A pair (v, ν) belongs to $V^{1,0}$ if $v \in T^{1,0}M$ and $\nu \in \Lambda^{0,1}(M)$. Writing $[(v,\nu),(v',\nu')]_d = ([v,v'], i_v(d\nu') - \operatorname{Lie}_{v'}\nu)$, we notice that $d\nu'$ is of Hodge type (0,2)+(1,1) by integrability of J, hence $i_v(d\nu')$ is of type (0,1), and $\operatorname{Lie}_{v'}\nu = \{i_{v'}, d\}\nu = i_{v'}(d\nu)$ is of type (0,1) for the same reason. ■

Pure spinors and generalized complex structures

REMARK: Let (V,h) be a vector space equipped with a scalar product, S the corresponding spinors, and $V\otimes S \longrightarrow S$ the Clifford multiplication. Given a non-zero spinor $\Psi\in S$, consider the space

$$\ker\Psi := \{v\in V \mid v\cdot\Psi = 0\}.$$

Then $\ker\Psi$ is isotropic. Indeed, for each $u,v\in\ker\Psi$, one has $0 = uv\Psi + uv\Psi = h(u,v)\Psi$.

DEFINITION: (Cartan, Chevalley)
$\Psi\in S$ is **a pure spinor** if $\ker\Psi$ is maximal isotropic.

THEOREM: (Chevalley) Let (V,h) be a vector space equipped with a scalar product, and S its spinor space. Then for each maximally isotropic subspace $W\subset V$, one has **$W=\ker\Psi$ for some pure spinor $\Psi\in S$,** which is **unique up to a scalar multiplier.**

Proof: Identifying V with $W\oplus W^*$, we obtain an identification $S=\Lambda^*W$ as above. Let $w_1,...,w_n$ be a basis in W. Then $\ker w_1\wedge w_2\wedge ...\wedge w_n = W$.

Converse is also obvious: if $\Psi\in\Lambda^*W$ satisfies $W\wedge\Psi = 0$, one has $\Psi = Cw_1\wedge w_2\wedge ...\wedge w_n$. ■

Pure spinors and maximally isotropic subspaces in $TM \oplus T^*M$

DEFINITION: Let $V = TM \oplus T^*M$, and $S = \Lambda^*(M)$ the corresponding spinor space. In this situation, **a pure spinor** is a nowhere vanishing differential form $\Psi \in \Lambda^*(M)$ such that the kernel $\ker\Psi$ of the Clifford multiplication $\Gamma_\Psi : V \longrightarrow \Lambda^*(M)$ has maximal possible dimension.

EXAMPLE: Let (M,J) be a complex n-manifold, and I the generalized complex structure constructed as in above. The corresponding pure spinor **is any non-degenerate section of $\Lambda^{n,0}(M,J)$. (check this).**

EXAMPLE: Let (M,ω) be a symplectic n-manifold, and I the generalized complex structure $I := \begin{pmatrix} 0 & -\omega \\ \omega^{-1} & 0 \end{pmatrix}$ as above. **The corresponding pure spinor is $\Psi = e^{\sqrt{-1}\,\omega}$.** Indeed, $V^{1,0}$ is spanned by $i_x - \sqrt{-1}\, e_{i_x(\omega)}$, for all $x \in TM$. Since i_x is a derivation, one has $i_x(e^{\sqrt{-1}\,\omega}) = \sqrt{-1}\, i_x(\omega) \wedge e^{\sqrt{-1}\,\omega}$, giving

$$i_x - \sqrt{-1}\, e_{i_x(\omega)}(e^{\sqrt{-1}\,\omega}) = \sqrt{-1}\, i_x(\omega) e^{\sqrt{-1}\,\omega} - \sqrt{-1}\, e_{i_x(\omega)}(e^{\sqrt{-1}\,\omega}) = 0.$$

Pure spinors and generalized complex structures

THEOREM: Let L be a maximal isotropic subbundle in $V = TM \oplus T^*M$, and $\Psi \in \Lambda^* M$ the corresponding spinor. Then L satisfies $[L, L]_d \subset L$ **if and only if $d\Psi = t\Psi$, for some $t \in V$.**

Proof. Step 1: If $u, v \in \ker \Psi$, then

$$[u, v]_d \Psi = [du + ud, v]\Psi = duv\Psi + udv\Psi - vdu\Psi - vud\Psi = -vud\Psi.$$

Since $u, v \in \ker \Psi$, this gives $[u, v]_d \Psi = -vud\Psi$. If $d\Psi = t\Psi$, one has $[u, v]_d \Psi = -vut\Psi = vtu\Psi - v(u, t)\Psi = 0$.

Step 2: It remains to show that $[L, L]_d \subset L$ implies that $d\Psi = t\Psi$, where Ψ is a pure spinor such that $L = \ker \Psi$. Consider the filtration on the spinor bundle $S = \Lambda^*(M)$, with $S_0 = \langle \Psi \rangle, S_1 = V \cdot \Psi, ..., S_d = V \cdot S_{d-1}$. Denote $L = \ker \Psi$ by $V^{1,0}$. Let $\Lambda^r V^{1,0} \subset Cl(V)$ be the subspace in the Clifford algebra generated by the monomials of degree r on $V^{1,0}$. Clearly,

$$S_r = \{s \in S \mid \Lambda^{r+1} V^{1,0} s = 0\}. \quad (*)$$

As shown above, $[L, L]_d \subset L$ is equivalent to $-vud\Psi = 0$ for all $u, v \in L$ By (*), $d\Psi \in S_1$ for all pure spinors Ψ inducing integrable generalized complex structure. However, $S_1 = V \cdot \Psi$. ∎

Pure spinors on symplectic manifolds

COROLLARY: If ω is a symplectic form, the generalized almost complex structure $\begin{pmatrix} 0 & -\omega \\ \omega^{-1} & 0 \end{pmatrix}$ is integrable. Indeed, **the corresponding spinor is e^{ω}, and it is closed as a differential form.**

DEFINITION: (Hitchin)
A **generalized Calabi-Yau manifold** is a generalized complex manifold, with the generalized complex structure defined by a pure spinor represented by a closed differential form.

EXAMPLE: Calabi-Yau manifolds and symplectic manifolds **are generalized Calabi-Yau.**