Unobstructed symplectic packing (1)

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Hyperkähler manifolds (reminder)

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Main result of lecture 2:

DEFINITION: A symplectic structure ω on a hyperkähler manifold is called **standard** if ω is a Kähler form for some hyperkähler structure.

REMARK: Any known symplectic structure on a hyperkähler manifold or a torus is of this type. **It was conjectured that non-standard symplectic structures don't exist.**

THEOREM: (E. Amerik, V.)
Let M be a maximal holonomy hyperkähler manifold. Then the period map $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is an open embedding on the set of all standard symplectic structures**, and **its image is the set of all cohomology classes v such that $q(\omega,\omega) > 0$,** where q is a quadratic form on cohomology defined below.

Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki) Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(b_2-3,3)$.** It is positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

Ergodicity of mapping class group action

THEOREM: (V., 2009)
Let M be a maximal holonomy hyperkähler manifold. **Then the image of the mapping class group Γ in $O(H^2(M,\mathbb{Z}))$ has finite index.**

COROLLARY: Γ acts on Teich_s **with dense orbits.**

Proof: We use a theorem of Calvin Moore:

THEOREM: (Calvin C. Moore, 1966) Let Γ be a lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact semisimple Lie subgroup. **Then the left action of Γ on G/H is ergodic.**

Applying this theorem to Γ inside $G = SO(H^2(M,\mathbb{R}),q)$ and H the stabilizer of $\omega \in H^2(M,\mathbb{R})$, we obtain that the action of Γ on $\operatorname{Teich}_s \subset H^2(M,\mathbb{R})$ **is ergodic on the set of symplectic form of a given volume**, hence has dense orbits. ■

COROLLARY: Any continuous invariant of symplectic structures is constant!

Let's look for such invariants!

Plan:

1. Symplectic packing and Gromov capacities.

2. Symplectic cut and symplectic blow-up

3. Full symplectic packing for hyperkähler manifolds

Full symplectic packing

DEFINITION: A **symplectic ball** is a ball of radius r in $\mathbb{R}^{2n}$, equipped with a standard symplectic structure $\omega = \sum dp_i \wedge dq_i$.

DEFINITION: Let M be a compact symplectic manifold of volume V. We say that M **admits a full, or unobstructed, symplectic packing** if for any disconnected union S of symplectic balls of total volume less than V, S admits a symplectic embedding to M.

DEFINITION: A symplectic structure ω on a torus is called **standard** if there exists a flat torsion-free connection preserving ω.

THEOREM: **(Latschev, McDuff, Schlenk, 2011)**
All 4-dimensional tori with standard symplectic structures admit full symplectic packing.

Main result

DEFINITION: Let M be a compact symplectic manifold of volume V. We say that M **admits a full, or unobstructed, symplectic packing** if for any disconnected union S of symplectic balls of total volume less than V, S admits a symplectic embedding to M.

DEFINITION: A symplectic structure ω on a torus is called **standard** if there exists a flat torsion-free connection preserving ω. A symplectic structure ω on a hyperkähler manifold is called **standard** if ω is a Kähler form for some hyperkähler structure.

REMARK: Any known symplectic structure on a hyperkähler manifold or a torus is of this type. **It was conjectured that non-standard symplectic structures don't exist.**

THEOREM: (M. Entov, V.)
Let M be a compact even-dimensional torus, or a hyperkähler manifold (such as a K3 surface), and ω a standard symplectic form. **Then (M,ω) admits a full symplectic packing.**

REMARK: In this talk, **all tori are compact, even-dimensional, and satisfy $\dim_{\mathbb{R}} M \geqslant 4$.**

Gromov Capacity

DEFINITION: Let M be a symplectic manifold. Define **Gromov capacity** $\mu(M)$ as the supremum of radii r, for all symplectic embeddings from a symplectic balls B_r to M.

DEFINITION: Define **symplectic volume** of a symplectic manifold (M, ω) as $\int_M \omega^{\frac{1}{2}\dim_{\mathbb{R}} M}$.

REMARK: Gromov capacity is obviously bounded by the symplectic volumes: a manifold of Gromov capacity r has volume $\geq \operatorname{Vol}(B_r)$. However, **there are manifolds of infinite volume with finite Gromov capacity.**

THEOREM: (Gromov)
Consider **a symplectic cylinder** $C_r := \mathbb{R}^{2n-2} \times B_r$ with the product symplectic structure. Then the Gromov capacity of C_r is r.

REMARK: This result was used by Gromov to study symplectic packing in $\mathbb{C}P^2$. He proved that **there is no full symplectic packing,** and found precise bounds.

Symplectic packing in $\mathbb{C}P^2$ (Gromov, McDuff, Polterovich, Biran)

THEOREM: Let v_N be a supremum of number V such that a collection of N equal symplectic balls of total volume V can be embedded to symplectic $\mathbb{C}P^2$ of volume 1. Then

N	1	2	3	4	5	6	7	8	9	$N > 9$
ν_N	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{20}{25}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1	1

The first few numbers are due to Gromov, last to Biran, the rest are McDuff-Polterovich.

REMARK: **These numbers are related to Nagata conjecture**, which is still unsolved (Biran used Taubes' work on Seiberg-Witten invariants to avoid proving it).

CONJECTURE: Suppose $p_1, ..., p_r$ are very general points in $\mathbb{C}P^2$ and that $m_1, ..., m_r$ are given positive integers. **Then for any $r > 9$ any curve C in $\mathbb{C}P^2$ that passes through each of the points p_i with multiplicity m_i must satisfy** $\deg C > \frac{1}{\sqrt{r}} \sum_{i=1}^r m_i$.

REMARK: Nagata conjecture was known already to Nagata when r is a full square, **and unknown for all other r, even when $p_1, ..., p_r$ are generic.**

Ekeland-Hofer theorem

THEOREM: (Ekeland-Hofer)
Let M, N be symplectic manifolds, and $\varphi: \ M \longrightarrow N$ a diffeomorphism. Suppose that for all sufficiently small, convex open sets $U \subset M$, Gromov capacity satisfies $\mu(U) = \mu(\varphi(U))$. **Then φ is a symplectomorphism.**

REMARK: This can be used to define C^0- **(continuous) symplectomorphisms.**

REMARK: Ekeland-Hofer theorem implies a theorem of Gromov-Eliashberg: **symplectomorphism group is C^0-closed in the group of diffeomorphisms.**

McDuff and Polterovich for Kähler manifolds

DEFINITION: Let M be a symplectic manifold, $x_1, ..., x_n \in M$ distinct points, and $r_1, ..., r_n$ a set of positive numbers. We say that M **admits symplectic packing** with centers $x_1, ..., x_n$ and radii $r_1, ..., r_n$ if there exists a symplectic embedding from a disconnected union of symplectic balls of radii $r_1, ..., r_n$ to M mapping centers of balls to $x_1, ..., x_n$.

THEOREM: (McDuff, Polterovich, 1995)
Let (M, ω) be a Kähler manifold, $\tilde{M} \xrightarrow{\nu} M$ its blow-up in $x_1, ..., x_n$, E_i the corresponding exceptional divisors, and $[E_i]$ their fundamental classes. Assume that the class $\nu^* \omega - \sum_i c_i [E_i]$ is Kähler, for some $c_i > 0$. **Then M admits a symplectic packing with radii $r_i = \pi^{-1} \sqrt{c_i}$.**

REMARK: **Converse is also true** (see below).

Complex manifolds (reminder)

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

Kähler manifolds (reminder)

DEFINITION: A Riemannian metric g on an almost complex manifold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M. The set of all Kähler classes is called **the Kähler cone**.

Kähler structure on a blow-up

DEFINITION: Let S be a total space of a line bundle $\mathcal{O}(-1)$ on $\mathbb{C}P^n$, identified with a space of pairs $(z \in \mathbb{C}P^n, t \in z)$, where t is a point on a line $z \subset \mathbb{C}^{n+1}$ representing z. The forgetful map $\pi: S \longrightarrow \mathbb{C}^{n+1}$ is called **a blow-up of $\mathbb{C}^{n+1}$ in 0**. Given an open ball $B \subset \mathbb{C}^{n+1}$, the map $\pi: \pi^{-1}(B) \longrightarrow B$ is called **a blow-up of B in 0**. To blow up a point in a complex manifold M, we remove a ball B around this point, and replace it with a blown-up ball $\tilde{B}$, gluing $B\backslash x \subset \tilde{B}$ with $B\backslash x \subset M$.

PROBLEM: Suppose that M is Kähler, and $\tilde{M}$ is its blow-up. **Find a Kähler metric on $\tilde{M}$ and write it explicitly.**

Answer: Symplectic blow-up!

REMARK: In this talk, I would often drop all π and other constants from the equations.

Symplectic quotient

DEFINITION: Let ρ be an S^1-action on a symplectic manifold (M,ω) preserving the symplectic structure, and $\vec{v}$ its unit tangent vector. Cartan's formula gives $0 = \operatorname{Lie}_{\vec{v}}\omega = d(\omega \lrcorner \vec{v})$, hence $\omega \lrcorner \vec{v}$ is a closed 1-form. **Hamiltonian**, or **moment map** of ρ is an S^1-invariant function μ such that $d\mu = \omega \lrcorner \vec{v}$, and **symplectic quotient** $M /\!\!/_c S^1$ is $\mu^{-1}(c)/S^n$.

REMARK: In these assumptions, restriction of the symplectic form ω to $\mu^{-1}(c)$ vanishes on $\vec{v}$, hence it is **obtained as a pullback of a closed 2-form $\omega_{/\!\!/}$ on $M /\!\!/_c S^1$.**

THEOREM: The form $\omega_{/\!\!/}$ is a symplectic form on $M /\!\!/_c S^1$. In other words, **the symplectic quotient is a symplectic manifold.**

REMARK: If, in addition, M is equipped with a Kähler structure (I,ω), and S^1-action preserves the complex structure, the symplectic quotient $M /\!\!/_c S^1$ inherits the Kähler structure. In this case it is called **a Kähler quotient**. Whenever the S^1-action can be integrated to holomorphic $\mathbb{C}^*$-action, **the Kähler quotient is identified with an open subset of its orbit space.**

REMARK: The moment map is defined by $d\mu = \omega \lrcorner \vec{v}$ uniquely up to a constant. However, the symplectic quotient $M /\!\!/_c S^1 = \mu^{-1}(c)/S^n$ **depends heavily on the choice of $c \in \mathbb{R}$.**

Symplectic blow-up

CLAIM: Consider the standard S^1-action on $\mathbb{C}^n$, and let $W \subset \mathbb{C}^n$ be an S^1-invariant open subset. Consider the product $V := W \times \mathbb{C}$ with the standard symplectic structure and take the S^1-action on $\mathbb{C}$ opposite to the standard one. **Then its moment map is $w - t$, where $w(x) = |x|^2$ is the length function on W and $r(t) = |t|^2$ the length function on $\mathbb{C}$.**

DEFINITION: **Symplectic cut** of W is $(W \times \mathbb{C}) /\!\!/_c S^1$.

REMARK: Geometrically, the symplectic cut is obtained as follows. Take $c \in \mathbb{R}$, and let $W_c := \{w \in W \mid |w|^2 \leqslant c\}$. Then W_c is a manifold with boundary ∂W_c, which is a sphere $|w|^2 = c$. Then $(W \times \mathbb{C}) /\!\!/_c S^1 = (W_c \times \mathbb{C}) /\!\!/_c S^1$ is obtained from W_c by gluing each S^1-orbit which lies on ∂W_c to a point. Combinatorially, $(W \times \mathbb{C}) /\!\!/_c S^1$ is $\mathbb{C}^n$ with 0 replaced with $\mathbb{C}P^{n-1}$.

DEFINITION: In these assumptions, **symplectic blow-up** of radius $\lambda = \sqrt{c}$ of W in 0 is $(W \times \mathbb{C}) /\!\!/_c S^1$. **Symplectic blow-up** of a symplectic manifold M is obtained by removing a symplectic ball W and gluing back a blown-up symplectic ball $(W \times \mathbb{C}) /\!\!/_c S^1$.

McDuff and Polterovich: symplectic packing from symplectic blow-ups

DEFINITION: Let M be a symplectic manifold, $x_1, ..., x_n \in M$ distinct points, and $r_1, ..., r_n$ a set of positive numbers. We say that M **admits symplectic packing** with centers $x_1, ..., x_n$ and radii $r_1, ..., r_n$ if there exists a symplectic embedding from a disconnected union of symplectic balls of radii $r_1, ..., r_n$ to M mapping centers of balls to $x_1, ..., x_n$.

REMARK: The choice of x_i is irrelevant, because **the group of symplectic authomorphisms acts on M infinitely transitively.**

Theorem 1: (McDuff-Polterovich)
Let (M, ω) be a symplectic manifold, $x_1, ..., x_n \in M$ distinct points, and $c_1, ..., c_n$ a set of positive numbers. Let $\pi : \tilde{M} \longrightarrow M$ be a symplectic blow-up with centers in x_i, and $E_i \in H^2(\tilde{M}, \mathbb{Z})$ the fundamental classes of its exceptional divisors. Then the following conditions are equivalent.

(i) M **admits a symplectic packing with radii** $r_i = \pi^{-1}\sqrt{c_i}$

(ii) For any $\alpha \in [0, 1]$, **there exists a form $\omega_\alpha(c_1, ..., c_n)$ cohomologically equivalent to $\pi^*\omega - \sum \alpha\pi c_i E_i$**, symplectic for $\alpha > 0$, smoothly depending on α, and satisfying $\omega_0(c_1, ..., c_n) = \pi^*\omega$. ■

McDuff and Polterovich for Kähler manifolds

REMARK: In Kähler situation, the smooth dependence condition is trivial, because for any two Kähler forms ω, ω', straight interval connecting ω to ω' consists of Kähler forms (indeed, **the set of Kähler forms is convex**). This brings the following corollary.

Corollary 1: Let (M, ω) be a Kähler manifold, $\tilde M \stackrel{\pi}{\longrightarrow} M$ its blow-up in $x_1, ..., x_n$, E_i the corresponding exceptional divisors, and $[E_i]$ their fundamental classes. Assume that the class $\pi^*\omega - \sum_i c_i[E_i]$ is Kähler, for some $c_i > 0$. **Then M admits a symplectic packing with radii $r_i = \pi^{-1}\sqrt{c_i}$.**

McDuff and Polterovich for tamed manifold

DEFINITION: An almost complex structure I on M **is tamed by a symplectic form** $\omega \in \Lambda^2 M$ if $\omega(x, Ix) > 0$ for any non-zero tangent vector $x \in TM$.

THEOREM: (McDuff-Polterovich, 1995) Let (M, ω) be a compact symplectic manifold, $\tilde{M} \xrightarrow{\nu} M$ its symplectic blow-up in $x_1, ..., x_n$, E_i the corresponding exceptional divisors, $[E_i]$ their fundamental classes, and $r_1, \ldots, r_k$ a collection of positive numbers. Assume there exists an almost complex structure I of on M tamed by ω and a symplectic form $\tilde{\omega}$ on $\tilde{M}$ taming the pullback almost complex structure $\tilde{I}$ so that $[\tilde{\omega}] = \nu^*[\omega] - \pi \sum_{i=1}^k r_i^2 [E_i]$. **Then (M, ω) admits a symplectic embedding of $\bigsqcup_{i=1}^{k} B^{2n}(r_i)$.** ■

Hyperkähler manifolds (reminder)

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

COROLLARY: The group $SU(2)$ of orthogonal quaternions acts on triples (I, J, K) producing new hyperkähler structures.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Calabi-Yau and Bogomolov decomposition theorem (reminder)

REMARK: A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

CLAIM: A compact hyperkähler manifold M **has maximal holonomy of Levi-Civita connection** $Sp(n)$ if and only if $\pi_1(M) = 0$, $h^{2,0}(M) = 1$.

THEOREM: (Bogomolov decomposition)
Any compact hyperkähler manifold has a finite covering isometric to a product of a torus and several maximal holonomy hyperkähler manifolds.

Campana simple manifolds

DEFINITION: A complex manifold M, $\dim_{\mathbb{C}} M > 1$, is called **Campana simple** if the union $\mathfrak{U}$ of all complex subvarieties $Z \subset M$ satisfying $0 < \dim Z < \dim M$ has measure 0. A point which belongs to $M \backslash \mathfrak{U}$ is called **generic**.

REMARK: **Campana simple manifolds are non-algebraic**. Indeed, a manifold which admits a globally defined meromorphic function f is a union of zero divisors for the functions $f - a$, for all $a \in \mathbb{C}$, and the zero divizor for f^{-1}. Hence **Campana simple manifolds admit no globally defined meromorphic functions**.

EXAMPLE: **A general complex torus has no non-trivial complex subvarieties,** hence it is Campana simple.

EXAMPLE: Let (M, I, J, K) be a hyperkähler manifold, and $L = aI + bJ + cK$, $a^2 + b^2 + c^2 = 1$ be a complex structure induced by quaternions. Then for all such (a, b, c) outside of a countable set, **all complex subvarieties $Z \subset (M, L)$ are hyperkähler, and (unless M a finite quotient of a product) $\bigcup_Z Z \neq M$** (V., 1994, 1996). **Therefore, (M, L) is Campana simple.**

CONJECTURE: (Campana)
Let M be a Campana simple Kähler manifold. **Then M is bimeromorphic to a finite quotient of a hyperkähler orbifold or a torus.**

Demailly-Paun theorem

REMARK: Let M be a compact Kähler manifold. Recall that the cohomology space $H^2(M,\mathbb{C})$ is decomposed as $H^2(M,\mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$ with $H^{1,1}(M)$ identified with the space of I-invariant harmonic 2-forms, and $H^{2,0}(M) \oplus H^{0,2}(M)$ the space of I-antiinvariant harmonic 2-forms. This decomposition is called **Hodge decomposition**. The space $H^{1,1}(M)$ is a complexification of a real space $H^{1,1}(M,\mathbb{R}) = \{\nu \in H^2(M,\mathbb{R}) \ \ | \ \ I(\nu) = \nu\}$.

THEOREM: (Demailly-Păun, 2002)
Let M be a compact Kähler manifold, and $\hat{K}(M) \subset H^{1,1}(M,\mathbb{R})$ a subset consisting of all (1,1)-forms η which satisfy $\int_Z \eta^k > 0$ for any k-dimensional complex subvariety $Z \subset M$. **Then the Kähler cone of M is one of the connected components of $\hat{K}(M)$.** ∎

Kähler cone for blow-ups of Campana simple manifolds

Theorem 2: Let M be a Campana simple compact Kähler manifold, and $x_1, ..., x_n$ distinct generic points of M. Consider the blow-up $\tilde{M}$ of M in $x_1, ..., x_n$, let E_i be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ its fundamental classes. Decompose $H^{1,1}(\tilde{M}, \mathbb{R})$ as $H^{1,1}(\tilde{M}, \mathbb{R}) = H^{1,1}(M, \mathbb{R}) \oplus \bigoplus \mathbb{R}[E_i]$. Assume that η_0 is a Kähler class on M. **Then for any $\eta = \eta_0 + c_i[E_i]$, the following conditions are equivalent.**

(i) η is Kähler on $\tilde{M}$.

(ii) all c_i are negative, and $\int_M \eta^{\dim_{\mathbb{C}} M} > 0$.

Proof of (ii) $\Rightarrow$ (i). Step 1:
All proper complex subvarieties of $\tilde{M}$ are either contained in E_i, or do not intersect E_i. The condition "η_0 is Kähler on M" implies $\int_Z \eta^k > 0$ for all subvarieties not intersecting E_i. Since $[E_i]$ restricted to E_i is $-[\omega_{E_i}]$, where ω_{E_i} is the Fubini-Study form, $c_i < 0$ implies that $\int_Z \eta^k > 0$ for all subvarieties which lie in E_i. Finally, the integral of η over $\tilde{M}$ is positive by the assumtion $\int_M \eta^{\dim_{\mathbb{C}} M} > 0$. **Therefore, the condition (ii) implies that $\eta \in \hat{K}(\tilde{M})$.**

Kähler cone for blow-ups of Campana simple manifolds (cont.)

Theorem 2: Let M be a Campana simple compact Kähler manifold, and $x_1, ..., x_n$ distinct generic points of M. Consider the blow-up $\tilde{M}$ of M in $x_1, ..., x_n$, let E_i be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ its fundamental classes. Decompose $H^{1,1}(\tilde{M}, \mathbb{R})$ as $H^{1,1}(\tilde{M}, \mathbb{R}) = H^{1,1}(M, \mathbb{R}) \oplus \bigoplus \mathbb{R}[E_i]$. Assume that η_0 is a Kähler class on M. **Then for any $\eta = \eta_0 + c_i[E_i]$, the following conditions are equivalent.**

(i) η is Kähler on $\tilde{M}$.

(ii) all c_i are negative, and $\int_M \eta^{\dim_{\mathbb{C}} M} > 0$.

Proof of (ii) ⇒ (i). Step 2:
The form η_0 is Kähler on M, hence it lies on the boundary of the Kähler cone of $\tilde{M}$, and η_0 can be obtained as a limit

$$\eta_0 = \lim_{\varepsilon \to 0} \eta_0 + \varepsilon c_i[E_i]$$

of forms which lie in the same connected component of $\hat{K}(\tilde{M})$. Therefore, η **belongs to the same connected component of $\hat{K}(\tilde{M})$ as a Kähler form.** By Demailly-Păun, this implies that η is Kähler.

Proof of (i) ⇒ (ii).
The numerical conditions of (ii) mean that $\eta \in \hat{K}(\tilde{M})$, hence they are satisfied automatically, as follows from Step 1. ■

Campana simple manifolds and symplectic packings

DEFINITION: Let M be a compact symplectic manifold of volume V. We say that M **admits a full symplectic packing** if for any disconnected union S of symplectic balls of total volume less than V, S admits a symplectic embedding to M.

Theorem 3: Let (M, I, ω_0) be a Kähler, compact, Campana simple manifold. **Then M admits a full symplectic packing.**

Proof. Step 1: Let $x_1, ..., x_n$ distinct generic points of M. Consider the blow-up $\tilde{M}$ of M in $x_1, ..., x_n$, let E_i be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ their fundamental classes. As follows from McDuff-Polterovich, existence of full symplectic packing on M is implied by existence of a Kähler form $\omega(c_1, ..., c_n)$ on $\tilde{M}$ with cohomology class $[\omega(c_1, ..., c_n)] = [\omega_0] - \sum c_i[E_i]$ for all $(c_1, ..., c_n)$ satisfying $\int_{\tilde{M}}([\omega_0] - \sum c_i[E_i])^n > 0$

Step 2: Such a form exists by Theorem 2. ■

Symplectic packing on hyperkähler manifolds and compact tori with irrational symplectic form

DEFINITION: A symplectic form is called **irrational** if its cohomology class is irrational, that is, lies in $H^2(M,\mathbb{R})\backslash\mathbb{R}\cdot H^2(M,\mathbb{Q})$.

THEOREM: Let M be a hyperkähler manifold or a compact torus, ω an irrational, standard symplectic form, and $\mathcal{T}$ the set of complex structures for which ω is Kähler. **Then the set $\mathcal{T}_0\subset\mathcal{T}$ of Campana simple complex structures is dense in $\mathcal{T}$ and has full measure in the corresponding moduli space.**

Proof: The Hodge loci of hyperkähler manifolds admitting a non-hyperkähler subvariety have positive codimension, and deformations of hyperkähler subarieties never cover M. ■

COROLLARY: Let M be a hyperkähler manifold or a compact torus, equipped with a standard, irrational symplectic form ω. **Then M admits full symplectic packing.**

Proof: By definition of a standard symplectic form, there exists a complex structure I such that ω is Kähler. Deforming I in $\mathcal{T}$, we obtain a Campana simple complex structure for which ω is Kähler. Then (M,ω) admits full symplectic packing by Theorem 3. ■

Symplectic cone and Kähler cone

DEFINITION: An almost complex structure I **tames** a symplectic structure ω if $\omega_I^{1,1}$ is a Hermitian form on (M, I).

PROPOSITION: Let (M, I, ω) be an almost complex tamed symplectic manifold, and $\eta \in \Lambda^{2,0+0,2}(M, \mathbb{R})$ a closed real $(2,0)+(0,2)$-form. **Then $\omega + \eta$ is also a symplectic form.** Moreover, **the complex structure I is tamed by $\omega + \eta$.**

Proof. Step 1: Since $I(\eta) = -\eta$ for each $(2,0)+(0,2)$-form η, one has $\omega(x, Ix) = \omega^{1,1}(x, Ix) > 0$ for each non-zero x.

Step 2: Since $\eta^{1,1} = 0$, one has $\omega + \eta(x, Ix) = \omega^{1,1}(x, Ix) > 0$ for each non-zero x. Therefore, $\omega + \eta$ is non-degenerate. ■

DEFINITION: A **symplectic class** of a manifold M is a cohomology class of a symplectic form on M. **Symplectic cone** of a symplectic manifold M is a set $\operatorname{Symp}(M) \subset H^2(M, \mathbb{R})$ of all symplectic classes. **Taming cone** of (M, I) is a cone of symplectic classes of all symplectic form taming an almost complex structure I.

Corollary 1: Let M be a Kähler manifold, and $\operatorname{Kah}(M)$ its Kähler cone. **Then the taming cone of M contains $\operatorname{Kah}(M) + H^{2,0+0,2}(M, \mathbb{R})$.** ■

Symplectic cone for blown-up tori and hyperkähler manifolds

Theorem 4: Let (M, I, ω_I) be a compact Kähler manifold obtained as a limit of Campana simple manifolds. **Then (M, ω_I) admits full symplectic packing.**

Proof. Step 1: Let B be an open neighbourhood of I in the moduli space of complex structures on M, and $B \stackrel{\varphi}{\longrightarrow} H^2(M, \mathbb{R})$ a map putting J to $(\omega_I)_J^{1,1}$. By Kodaira stability theorem, $\varphi(J)$ is a Kähler class for J sufficiently close to I. Therefore, **there exists a Campana simple complex structure J such that $\omega_J := (\omega_I)_J^{1,1}$ is Kähler, arbitrarily close to I in B.**

Step 2: By Theorem 3, $\eta_J := \omega_J + \sum c_i[E_i]$ **is a Kähler class on a blow-up of (M, J), with blow-up points generic.** Indeed, the condition $\int_M \eta_J^{\dim_{\mathbb{C}} M} > 0$ remains true for J sufficiently close to I.

Step 3: Now, $\omega_I - \omega_J$ is by definition a $(2,0)+(0,2)$-cohomology class on (M, J). Therefore, $\eta = \omega_I + \sum c_i[E_i]$ is obtained from a Kähler form η_J by adding a $(2,0)+(0,2)$-form on (M, J). This implies that η **belongs to taming cone,** and Theorem 4 follows from the taming version of McDuff-Polterovich.

■

Further directions

1. We explored symplectic packing by symplectic balls. What about a packing by other subsets $K \subset \mathbb{R}^{2n}$?

1A. Define a packing number $\nu(K, M)$ of (K, ω) to M as a supremum of all ε for which $(K, \varepsilon\omega)$ admits a symplectic embedding to M. This function is obviously semicontinuous on K and M. When K is a union of symplectic balls, and M a hyperkähler manifold or a torus, $\nu(K, M) = \frac{\operatorname{Vol}(M)}{\operatorname{Vol}(K)}$. Using ergodicity, it is possible to show that $\frac{\nu(K,M)}{\operatorname{Vol}(M)}$ is constant for irrational symplectic structures on such M. Is it equal to 1? If so, we have "full packing by K".

2. Replacing blow-ups by orbifold blow-ups and balls by symplectic ellipsoids with rational axis length, our argument would give full packing by ellipsoids (paper in preparation, jointly with M. Entov).

3. Let Symp be the infinite-dimensional Frechet manifold of all symplectic forms on M, and Diff the diffeomorphism group. The full packing phenomena seems to be related to ergodicity of Diff-action on Symp: the packing defines a semi-continuous, Diff-invariant function on Symp, which should be a posteriori constant on the set of all symplectic structures with dense Diff-orbits. One could study other semi-continuous quantities in relation to Diff-action and ergodicity.