Teichmüller spaces and moduli of geometric structures (1)

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Plan:

1. Set-up: Teichmüller space and the moduli space of geometric structures.

2. Moser's theorem. Teichmüller space of symplectic structures on a torus.

3. Other geometric structures and their Teichmüller spaces. G_2 structures, holomorphically symplectic structures.

Geometric structures

DEFINITION: "Geometric structure" on a manifold M is a reduction of its structure group $GL(n,\mathbb{R})$ to a subgroup $G \subset GL(n,\mathbb{R})$. However, it is easier to define it by a collection of tensors $\Psi_1, ..., \Psi_n$ such that the stabilizer $\operatorname{St}_{\langle\Psi_1,...,\Psi_n\rangle}$ of $\Psi_1, ..., \Psi_n$ at each point $x \in M$ is conjugate to $G \subset GL(T_xM)$.

Let me give some examples.

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: Symplectic form on a manifold is a non-degenerate differential 2-form ω satisfying $d\omega = 0$.

Teichmüller space of geometric structures

Let $\mathcal{C}$ be the set of all geometric structures of a given type, say, complex, or symplectic. We put topology of uniform convergence with all derivatives on $\mathcal{C}$. Let $\operatorname{Diff}_0(M)$ be the connected component of its diffeomorphism group $\operatorname{Diff}(M)$ (**the group of isotopies**).

DEFINITION: The quotient $\mathcal{C}/\operatorname{Diff}_0$ is called **Teichmüller space** of geometric strictures of this type.

DEFINITION: The group $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ is called **the mapping class group** of M. It acts on Teich by homeomorphisms.

DEFINITION: The orbit space $\mathcal{C}/\operatorname{Diff} = \operatorname{Teich}/\Gamma$ is called **the moduli space** of geometric structure of this type.

Today I will describe Teich and Γ in some interesting cases and explain some important concepts, such as **ergodicity of Γ-action.**

Teichmüller space for symplectic structures

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\mathsf{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ is a Frechet vector space, and Symp a Frechet manifold.

DEFINITION: Consider the group of diffeomorphisms, denoted Diff or Diff(M) as a Frechet Lie group, and denote its connected component ("group of isotopies") by Diff_0. The quotient group $\Gamma := \mathrm{Diff} / \mathrm{Diff}_0$ is called **the mapping class group** of M.

DEFINITION: **Teichmüller space of symplectic structures on M** is defined as a quotient $\mathrm{Teich}_s := \mathrm{Symp} / \mathrm{Diff}_0$. The quotient $\mathrm{Teich}_s / \Gamma = \mathrm{Symp} / \mathrm{Diff}$, is called **the moduli space of symplectic structures.**

REMARK: In many cases Γ acts on Teich_s with dense orbits, hence **the moduli space is not always well defined.**

DEFINITION: Two symplectic structures are called **isotopic** if they lie in the same orbit of Diff_0, and **diffeomorphic** is they lie in the same orbit of Diff.

Moser's theorem

DEFINITION: Define **the period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

Theorem 1: (Moser)
Let ω_t, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold S. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are diffeomorphic.**

The proof of Moser's theorem

THEOREM: (Moser)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\text{Per}: \text{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

Proof. Step 1: We can locally find a section S for the Diff_0-action on Symp, producing a local decomposition $\text{Symp} = O \times S$, where O is a Diff_0-orbit. Here O and S are both Frechet manifolds.

Step 2: The period map $P: U \longrightarrow H^2(M, \mathbb{R})$ is a smooth submersion. By Theorem 1, the fibers of P are 0-dimensional. Therefore, P is locally a diffeomorphism. ■

Symplectic structures on a compact torus

DEFINITION: A symplectic structure ω on a torus is called **standard** if there exists a flat torsion-free connection preserving ω.

REMARK: Moser's theorem immediately implies that **the set** Teich_{st} **of standard symplectic structures is open in the Teichmüller space.** Indeed, the period map from Teich_{st} to $H^2(M)$ is also locally a diffeomorphism.

REMARK: It is not known if any non-standard symplectic structures exist (even in dimension =4).

THEOREM: Let $\Lambda^2_{nd}(H_1(T)) \subset H^2(T)$ be the space of symplectic forms on $H_1(T)$, where T is an even-dimensional torus. Consider the period map $\text{Per}: \text{Teich}_{st} \longrightarrow \Lambda^2_{nd}(H_1(T)) \subset H^2(T)$, where Teich_{st} is the Teichmüller space of standard symplectic structures on T. **Then** Per **is a diffeomorphism.**

The space of flat Hermitian metrics

THEOREM: Let $\Lambda^2_{nd}(H_1(T)) \subset H^2(T)$ be the space of symplectic forms on $H_1(T)$, where T is an even-dimensional torus. Consider the period map $\mathsf{Per}: \mathsf{Teich}_{st} \longrightarrow \Lambda^2_{nd}(H_1(T)) \subset H^2(T)$, where Teich_{st} is the Teichmüller space of standard symplectic structures on T. **Then Per is a diffeomorphism.**

Proof. Step 1: Let Teich_h be the Teichmüller space of flat Hermitian metrics on T. Clearly, $\mathsf{Teich}_h = GL(2n, \mathbb{R})/U(n)$. Moreover, **the natural forgetful map $\mathsf{Teich}_h \longrightarrow \mathsf{Teich}_{st}$ is surjective**.

Step 2: The fibers of the natural projection $\mathsf{Teich}_h \longrightarrow \Lambda^2_{nd}(H_1(T))$ are connected. Using the diagram

$$\begin{array}{ccc} \mathsf{Teich}_h & \longrightarrow & \Lambda^2_{nd}(H_1(T)) \\ \downarrow & & \downarrow \mathrm{Id} \\ \mathsf{Teich}_{st} & \longrightarrow & \Lambda^2_{nd}(H_1(T)) \end{array}$$

we obtain that $\mathsf{Per}: \mathsf{Teich}_{st} \longrightarrow \Lambda^2_{nd}(H_1(T))$ has connected fibers. By Moser's theorem, this map is a diffeomorphism. ■

G_2-structures

DEFINITION: Let $\rho \in \Lambda^2\mathbb{R}^7$ be a 3-form on $\mathbb{R}^7$. We say that ρ is **non-degenerate** if the dimension of its stabilizer is maximal:

$$\dim St_{GL(7)}\rho = \dim GL(7) - \dim \Lambda^3(\mathbb{R}^7) = 49 - 35 = 14.$$

In this case, $St(\rho)$ is one of two real forms of a 14-dimensional Lie group $G_2(\mathbb{C})$. We say that ρ is **non-split** if it satisfies $St(\rho|_x) \cong G_2$, where G_2 denotes the compact real form of $G_2(\mathbb{C})$. **A G_2-structure** on a 7-manifold is a 3-form $\rho \in \Lambda^3(M)$, which is non-degenerate and non-split at each point $x \in M$ ("stable", in the sense of Hitchin).

REMARK: A form ρ defines a $\Lambda^7 M$-valued metric on M:

$$g(x,y) = (\rho \lrcorner x) \wedge (\rho \lrcorner y) \wedge \rho$$

This defines a conformal structure on M. The conformal factor is fixed if we want $|\rho| = 1$. Therefore, **every G_2-manifold is equipped with a natural Riemannian structure.**

Holonomy G_2-manifolds

DEFINITION: An G_2-manifold is called **a holonomy G_2-manifold** if ρ is preserved by the corresponding Levi-Civita connection.

REMARK: (M, ρ) **is a holonomy G_2-manifold if and only if the form ρ and its Hodge dual $*\rho$ are closed.**

THEOREM: (Joyce) Let $\operatorname{Teich}_{G_2}$ be the Teichmüller space of G_2-structures on a compact 7-manifold M, and $\operatorname{Per} \operatorname{Teich}_{G_2} \longrightarrow H^3(M)$ the period map associating to ρ its cohomology class. **Then Per is locally a diffeomorphism.**

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I : TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

Holomorphic symplectic form

DEFINITION: Holomorphic symplectic form on an almost complex manifold (M, I) is a non-degenerate differential 2-form $\omega \in \Lambda^2(M, \mathbb{C})$ satisfying $d\omega = 0$ and $\Omega(Ix, y) = \sqrt{-1}\,\Omega(x, y)$.

REMARK: This is the same as to say that $\Omega(X, \cdot) = 0$ **for all** $X \in T^{1,0}(M)$**, and** Ω **is non-degenerate on** $T^{0,1}(M)$**.**

DEFINITION: Let Ω be a differential form on M. The **kernel**, or **the null-space** $\ker(\Omega) \subset TM$ of Ω is the space of all vector fields $X \in TM$ such that the contraction $\Omega \lrcorner X$ vanishes.

Proof. Step 1: Let $X, X_1 \in \ker(\Omega)$, and $X_2, ..., X_p$ any vector fields. Cartan's formula implies that $\operatorname{Lie}_X(\Omega) = d(i_X(\Omega)) + i_x(d\Omega) = 0$, hence $\operatorname{Lie}_X(\Omega) = 0$.

Step 2: $\operatorname{Lie}_X(\Omega)(X_1, ..., X_p) = \operatorname{Lie}_X(\Omega(X_1, ..., X_p)) - \sum_{i=1}^p \Omega(X_1, ..., [X, X_i], ...X_p)$. All terms of this sum, except $\Omega([X, X_1], X_2, ..., X_p)$, vanish, because $X_1 \in \ker(\Omega)$. Since $\operatorname{Lie}_X(\Omega) = 0$, we have $\Omega([X, X_1], X_2, ..., X_p) = 0$ for all $X_2, ..., X_p$. Therefore, $[X, X_1] \in \ker(\Omega)$. ■

COROLLARY: Let (M, I) be an almost complex manifold admitting a holomorphic symplectic form. **Then I is integrable.**

Teichmüller space for holomorphic symplectic structures

THEOREM: (Kaledin, V.) Let (M,Ω) be a holomorphic symplectic manifold, $\operatorname{Teich}_\Omega$ the Teichmüller space of holomorphic symplectic structures on M, and $\operatorname{Per}: \operatorname{Teich}_\Omega \longrightarrow H^2(M,\mathbb{C})$ the map associating to (M,Ω) the cohomology class of Ω, **Then Per is locally an embedding.**