Limits of hyperkähler metrics

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What are the limits of Ricci-flat metrics on Calabi-Yau manifolds?

Let (M, I) be a holomorphically symplectic manifold with maximal holonomy and Picard group of non-maximal rank, and V_I the set of all Ricci-flat Kähler metrics on (M, I). By Calabi-Yau theorem, V_I is identified with the Kähler cone of (M, I), **which is $b_2 - 2$-dimensional.**

Let V be the set of all Ricci-flat Kähler metrics on M obtained as deformations of $h \in V_I$. **This set has real dimension $3b_2 - 5$.**

The main result of today's lecture.

THEOREM: All $h \in V$ can be obtained as Gromov-Hausdorff limits of the sequences $\{h_i\} \in V_I$.

Limits of hyperkähler metrics

Couple of simple observations.

1. The boundary of the Kähler cone of (M, I) in the Gromov space of all metric spaces seem to be **some sort of a strange fractal**, much bigger dimension than the cone itself. It is not clear if we have other (even more strange) metric spaces in this boundary, the dimension is also unknown.

2. The limits of the Ricci-flat Kähler forms in the space of currents are quite well-behaved.

3. These results **are also true for a torus**, where they can be accessed geometrically.

Plan:

1. State global Torelli theorem.

2. Explain the ergodic nature of the mapping group action.

3. Deduce the main theorem from the ergodicity phenomenon.

Teichmüller spaces

DEFINITION: Let M be a smooth manifold. **A complex structure** on M is an endomorphism $I \in \operatorname{End} TM$, $I^2 = -\operatorname{Id}_{TM}$ such that the eigenspace bundles of I are **involutive**, that is, satisfy satisfy $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

REMARK: Let Comp be the space of such tensors equipped with a topology of convergence of all derivatives. **It is a Fréchet manifold**.

REMARK: The diffeomorphism group Diff is a Fréchet Lie group acting on a Fréchet manifold Comp in a natural way.

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Let $\operatorname{Teich} := \operatorname{Comp} / \operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

REMARK: The set of equivalence classes of complex structures is identified with $\operatorname{Teich} / \Gamma$, where Γ is **the mapping class group**, $\Gamma = \operatorname{Diff} / \operatorname{Diff}_0$.

Holomorphically symplectic manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: Let $\omega_I, \omega_J, \omega_K$ be the Kähler symplectic forms associated with I, J, K. **A hyperkähler manifold is holomorphically symplectic:** $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I). **Converse is also true:**

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A compact hyperkähler manifold M is called **maximal holonomy manifold**, or **simple**, or **IHS** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: (Bogomolov, 1974) Any hyperkähler manifold admits a finite covering which is a product of a torus and several maximal holonomy (simple) hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

Computation of the mapping class group

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$,** and $c>0$ a rational number.

DEFINITION: The form q is called **Bogomolov-Beauville-Fujiki form**. It has signature $(3, b_2-3)$. The sign is chosen in such a way that q **is positive on the triple of Kähler forms $\omega_I, \omega_J, \omega_K$, and negative on their orthogonal complement.**

THEOREM: (V., 1996, 2009) Let M be a simple hyperkähler manifold, and $\Gamma_0 = \operatorname{Aut}(H^*(M,\mathbb{Z}))$. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

The period map

REMARK: To simplify the language, we redefine Teich and Comp for hyperkähler manifolds, restricting to complex structures of Kähler type and admitting a holomorphic symplectic form. Since the Hodge numbers are constant in families of Kähler manifolds, **for any $J \in$ Teich, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.**

Definition: Let Per : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map Per : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: Per **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}\mathrm{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$**.** Indeed, the group $SO(H^2(M, \mathbb{R}), q) = SO(b_2 - 3, 3)$ acts transitively on $\mathbb{P}\mathrm{er}$, and $SO(2) \times SO(b_2 - 3, 1)$ is a stabilizer of a point.

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts, 2001) Two points $I, I' \in \operatorname{Teich}$ **are non-separable if and only if there exists a bimeromorphism $(M, I) \longrightarrow (M, I')$ which is non-singular in codimension 2 and acts as identity on $H^2(M)$.**

REMARK: This is possible only if (M, I) and (M, I') contain a rational curve. **General hyperkähler manifold has no curves;** ones which have belong to a countable union of divisors in Teich.

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: (V., 2009) **The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism,** for each connected component of Teich_b.

Ergodic complex structures

DEFINITION: Let (M, μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.** ■

DEFINITION: A complex structure $I \in$ Teich is called **ergodic** if its orbit is dense in its connected component in Teich.

CLAIM: Let (M, I) be a manifold with an ergodic complex structure, and I' its deformation. **Then there exists a sequence of diffeomorphisms ν_i such that $\nu_i(I)$ converges to I' in C^∞-topology.** Moreover, this property is equivalent to ergodicity of I.

THEOREM: Let M be a compact torus, $\dim_{\mathbb{C}} M \geqslant 2$, or a simple hyperkähler manifold. **A complex structure on M is ergodic if and only if $\operatorname{Pic}(M)$ is not of maximal rank.**

Gromov-Hausdorff metrics

DEFINITION: Let $X \subset M$ be a subset of a metric space, and $y \in M$ a point. **Distance from a y to X** is $\inf_{x \in X} d(x, y)$. **Hausdorff distance** $d_H(X, Y)$ between to subsets $X, Y \subset M$ of a metric space is $\max(\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X))$. **Gromov-Hausdorff distance** between complete metric spaces X, Y of diameter $\leqslant d$ is an infimum of $d_H(\varphi(X), \psi(Y))$ taken over all isometric embeddings $\varphi: X \longrightarrow Z$, $\psi: Y \longrightarrow Z$ to a third metric space.

REMARK: This definition puts the structure of a metric space on the set of equivalence classes of all separable metric spaces.

REMARK: Let $\varphi: X \longrightarrow Y$ be a map of metric spaces (not necessarily continuous). Its **defect** δ_φ is $\inf_{x_1, x_2 \in X} |d(x_1, x_2) - d(\varphi(x_1), \varphi(x_2))|$. Gromov-Hausdorff distance between metric spaces X, Y is bounded (both directions) by the quantity $\hat{d}_{GH}(X, Y) = \inf_{\varphi, \psi} \max(\delta_\psi, \delta_\psi)$, where infimum is taken over the set of all maps $\varphi: X \longrightarrow Y$, $\psi: Y \longrightarrow X$:

$$C_1 \hat{d}_{GH}(X, Y) \leqslant d_{GH}(X, Y) \leqslant C_2 \hat{d}_{GH}(X, Y)$$

REMARK: This means that **a converging sequence of Riemannian metrics converges in Gromov-Hausdorff topology.**

Gromov's compactness theorem

DEFINITION: A subset $X \subset M$ is called **precompact** if its closure in M is compact.

DEFINITION: We say that **Ricci curvature of a Riemannian manifold (M,g) is bounded from below by c** if the symmetric form $\operatorname{Ric}_g - cg \in \operatorname{Sym}^2 T*M$ is positive definite.

THEOREM: (Gromov's compactness theorem)
Let W_d be the Gromov's space of all metric spaces of diameter d, and $X_{c,d} \subset W_d$ the space of all Riemannian manifolds with Ricci curvature bounded from below by c. **Then $X_{c,d}$ is precompact.**

QUESTION: Let Hyp_d be the space of all hyperkähler metrics of diameter d considered as a subset in W_d. **What is the shape of Hyp_d and its closure?**

Space of hyperkähler metrics and its closure

Related question: Consider hyperkähler forms ω_I as currents on (M, I), and let $\overline{\mathrm{Hyp}}_{cur}$ be its closure in the space of currents. **Is it related by $\overline{\mathrm{Hyp}}_d$?**

CONJECTURE: (some examples proven by Dinh-Sibony)
For any sequence of Ricci-flat metrics g_i on (M, I) such that the cohomology class of their symplectic forms converges to $[\omega_0] \in H^{1,1}(M, I)$, **the sequence ω_i converges to a unique positive current.**

Gromov-Hausdorff convergence is entirely different.

THEOREM: Let (M, I) be a hyperkähler manifold with ergodic complex structure (that is, non-maximal Picard number), and $\overline{V}_I$ the Gromov-Hausdorff closure of the space V_I of all hyperkähler metrics on M. **Then $\overline{V}_I$ contains the space Hyp of all hyperkähler metrics on M obtained by deformation from V_I.**

REMARK: $\dim V_I = b_2 - 2$, and $\dim \mathrm{Hyp} = 3b_2 - 5$: much bigger!

Kodaira stability theorem and its application

THEOREM: (Kodaira stability)
Let (M, J, ω) be a closed Kähler manifold, and let $\{J_t\}$, $t \in B$, $J_0 = J$, be a smooth local deformation of J. **Then there exists a neighborhood of $U \subset B$ of zero in B such that the complex structure J_t on M is of Kähler type and $[\omega]^{1,1}_{J_t}$ is a Kähler class for all $t \in U$.** ■

THEOREM: Let (M, I) be a hyperkähler manifold with ergodic complex structure (that is, non-maximal Picard number), and $\overline{V}_I$ the Gromov-Hausdorff closure of the space V_I of all hyperkähler metrics of diameter d on M. **Then $\overline{V}_I$ contains the space Hyp_d of all hyperkähler metrics on M obtained by deformation from V_I.**

Proof. Step 1: Let $g \in \mathrm{Hyp}_d$ be a hyperkähler metric on (M, J), and ω its Kähler form. Since I is ergodic, there exists a sequence $I_i \in Teich$ converging to $J \in \mathrm{Teich}$ such that $I_i = \nu_i(I)$.

Step 2: By Kodaira's stability, for i sufficiently big, there exists a Kähler class $[\omega_i]$ on (M, I_i) converging to $[\omega]$.

Step 3: Let g_i be the hyperkähler metric on (M, I_i) associated with the Kähler class $[\omega_i]$. Then (M, g_i) converges to (M, g) in C^∞-topology, because the Calabi-Yau metric depends C^∞-continuously on $(I, [\omega])$. ■