Hypercomplex Structures on Kähler Manifolds

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Estruturas geométricas em variedades
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IMPA

Almost complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

Kähler manifolds

DEFINITION: A Riemannian metric g on a complex manifold (M, I) is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

REMARK: This is equivalent to $\nabla\omega = 0$, where ∇ is Levi-Civita connection.

REMARK: Since restriction of a closed form is closed, **a complex submanifold of a Kähler manifold is again Kähler.**

Examples of Kähler manifolds.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **the Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n = \mathbb{C}^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: **Fubini-Study form is Kähler.** Indeed, $d\omega\Big|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish, because $-\operatorname{Id} \in U(n)$

REMARK: **The same argument works for all symmetric spaces.**

Corollary: **Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler.** Indeed, a restriction of a closed form is again closed.

Hyperkähler manifolds

DEFINITION: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators $I,J,K: TM \longrightarrow TM$, satisfying the quaternionic relation $I^2=J^2=K^2=IJK=-\operatorname{Id}$. Suppose that I, J, K are Kähler. Then (M,I,J,K,g) is called **a hyperkähler manifold**.

LEMMA: Let (M,I,J,K) be hyperkähler. **Then the form $\Omega := \omega_J + \sqrt{-1}\omega_K$ is a holomorphic symplectic 2-form on (M,I).**

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

THEOREM: (E. Calabi, 1952, S.-T. Yau, 1978)
Let M be a compact, holomorphically symplectic manifold admitting a Kähler metric. **Then M admits a hyperkähler metric, which is uniquely determined by the cohomology class of its Kähler form.**

HYPERCOMPLEX MANIFOLDS

a. k. a "Hyperkähler manifolds without a metric"

DEFINITION: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$. Suppose that I, J, K are integrable almost complex structures. Then (M, I, J, K) is called **a hypercomplex manifold**.

EXAMPLES:

1. In dimension 1 (real dimension 4), we have **a complete classification of compact hypercomplex manifolds,** due to C. P. Boyer (1988).

2. **Many homogeneous examples,** due to D. Joyce and physicists Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen (1980-ies, early 1990-ies).

3. Some nilmanifolds and solvmanifolds admit **locally homogeneous hypercomplex structure** (M. L. Barberis, I. Dotti, A. Fino) (1990-ies).

4. Some **inhomogeneous examples** are constructed by deformation or as fiber bundles.

In dimension > 1, **no classification results are known** *(and no conjectures either).*

OBATA CONNECTION

Hypercomplex manifolds can be characterized in terms of holonomy

THEOREM: (M. Obata, 1952) Let (M, I, J, K) be a hypercomplex manifold. **Then M admits a unique torsion-free affine connection preserving I, J, K.**

Converse is also true. Suppose that I, J, K are operators defining quaternionic structure on TM, and ∇ a torsion-free, affine connection preserving I, J, K. Then I, J, K are integrable almost complex structures, and (M, I, J, K) is hypercomplex.

Holonomy of Obata connection lies in $GL(n, \mathbb{H})$. Conversely, **a manifold equipped with an affine, torsion-free connection with holonomy in $GL(n, \mathbb{H})$ is hypercomplex**.

This can be used as a definition of a hypercomplex structure.

QUESTIONS

1. Given a complex manifold M, when M admits a hypercomplex structure? How many?

2. What are possible holonomies of Obata connection, for a compact hypercomplex manifold? Can $SL(n, \mathbb{H})$ be a local holonomy group of a compact manifold? $GL(n, \mathbb{H})$ can (Andrey Soldatenkov, 2012).

3. Describe the structure of automorphism group of a hypercomplex manifold.

The main result of today's talk.

THEOREM: Let (M, I, J, K) be a compact hypercomplex manifold. Assume that the complex manifold (M, I) admits a Kähler structure. **Then (M, I) is hyperkähler.**

Quaternionic Hermitian structures

DEFINITION: Let (M,I,J,K) be a hypercomplex manifold, and g a Riemannian metric. We say that g is **quaternionic Hermitian** if I,J,K are orthogonal with respect to g.

REMARK: A quaternionic Hermitian can be obtained as follows: **take any Riemannian metric, and average it with respect to I,J,K.**

CLAIM: Given a quaternionic Hermitian metric g on (M,I,J,K), consider its Hermitian forms $\omega_I(\cdot,\cdot) = g(\cdot, I\cdot), \omega_J, \omega_K$ (real, but **not closed**). **Then $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ is $\mathbb{C}$-linear with respect to I.**

REMARK: Denote the space of $\mathbb{C}$-linear p-forms on (M,I) by $\Lambda^{p,0}(M,I)$. Then **complex linearity of Ω can be written as $\Omega \in \Lambda^{2,0}(M,I)$.**

REMARK: This argument also implies that $c_1(M,I) = 0$. Indeed, the top exterior power of Ω is a non-degenerate section of the canonical bundle $\Lambda^{2n,0}(M,I)$, and $c_1(M,I) = c_1(\Lambda^{2n,0}(M,I))$.

HKT manifolds

REMARK: Let (M, I, J, K, g) be a quaternionic Hermitian manifold, and $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ the corresponding (2,0)-form, **and** $d\Omega = 0$, (M, I, J, K, g) **is hyperkähler** (this is one of the definitions). **Consider a weaker condition:** $\partial\Omega = 0$. Here, $\partial : \Lambda^{2,0}(M, I) \longrightarrow \Lambda^{3,0}(M, I)$ is the **"Dolbeault differential":** de Rham differential restricted to complex linear forms).

DEFINITION: (Howe, Papadopoulos, 1998)
Let (M, I, J, K) be a hypercomplex manifold, g a quaternionic Hermitian metric, and $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ the corresponding $(2, 0)$-form. We say that g is **HKT ("weakly hyperkähler with torsion")** if $\partial\Omega = 0$.

REMARK: **This definition is due to Grantcharov and Poon,** the definition of Howe-Papadopoulos is given in terms of the Bismut connections.

HKT-metrics play in hypercomplex geometry the same role as Kähler metrics play in complex geometry.

HKT manifolds with trivial canonical bundle

DEFINITION: The bundle of $(p,0)$-forms on a complex manifold (M,I), $\dim_{\mathbb{C}} M = d$ has a natural holomorphic structure. The line bundle $\Lambda^{d,0}(M,I)$ is called **canonical bundle**, or **canonical class**.

1. **HKT metrics admit a smooth potential (locally).** There is a notion of an "HKT-class" (similar to Kähler class) in a certain finite-dimensional coholology group. **Two metrics in the same HKT-class differ by $\partial J\bar\partial J$ of a a potential, which is a function.**

2. The canonical bundle $\Lambda^{2n,0}(M,I)$ of (M,I) is topologically trivial, but When (M,I) has (holomorphically) trivial canonical bundle, **a version of Lefschetz-type identities can be proven giving an $\mathfrak{sl}(2)$-action on cohomology $H^*(M,\mathcal{O}_{(M,I)}$.**

REMARK: Using the Calabi-Yau theorem, it is possible to show that **any compact Kähler manifold (M,I) with $c_1(M,\mathbb{Z})=0$ has holomorphically trivial canonical bundle.** There is a more complicated argument, due to F. Bogomolov, proving this result without the Calabi-Yau theorem. However, **it all fails for compact non-Kähler manifolds** (or non-compact non-Kähler).

Lefschetz identities for Kähler manifolds.

The usual Lefschetz $\mathfrak{sl}(2)$-action is constructed as follows. Let M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, ω its Kähler form, $L: \Lambda^i(M) \longrightarrow \Lambda^{i+2}(M)$ the operator of multiplication by ω, Λ its Hermitian adjoint, and H acts on $\Lambda^i(M)$ as a scalar multiplication by $(i-n)$. **Then (L, Λ, H) is an $\mathfrak{sl}(2)$-triple.** It commutes with the Laplacian, giving an **$\mathfrak{sl}(2)$-action on cohomology.**

Main ingredients of the proof:

0. **Kähler identities**, a. k. a. "supersymmetry": identities in the Lie superalgebra $\mathfrak{a} \subset \operatorname{End}(\Lambda^*(M))$ generated by the Dolbeault differential, its complex conjugate, their Hermitian adjoint operators, and the $\mathfrak{sl}(2)$-action.

1. **Use the linear algebra to show that (L, Λ, H) is an $\mathfrak{sl}(2)$-triple.** This is true for any almost complex Hermitian manifold

2. **Show that (L, Λ, H) commutes with the Laplacian.** Need Kähler identities for this.

Lefschetz identities for HKT-manifolds with trivial canonical bundle.

THEOREM: Let (M, I, J, K, g) be a hypercomplex manifold, $\dim_{\mathbb{H}}(M) = n$. Assume that the canonical bundle of $M_I := (M, I)$, is trivial as a holomorphic vector bundle. Consider the Dolbeault resolution for the holomorphic cohomology $H^*(M_I, \mathcal{O})$

$$\Lambda^0(M_I) \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M_I) \xrightarrow{\overline{\partial}} \Lambda^{0,2}(M_I) \xrightarrow{\overline{\partial}} \dots \quad (*)$$

The multiplication map $L(\eta) = \eta \wedge \overline{\Omega}$ commutes with the differential, because $\overline{\Omega}$ is $\overline{\partial}$-closed. Let Λ be its Hermitian adjoint, and $H(\eta) = i - n$, for all $\eta \in \Lambda^{0,i}(M_I)$. **Then (L, Λ, H) is an $\mathfrak{sl}(2)$-triple acting on cohomology of (*).**

REMARK: This resolution **computes the cohomology $H^*(M_I, \mathcal{O})$ of the sheaf of holomorphic functions on (M, I).**

REMARK: From this theorem we immediately obtain that **the cohomology class of $\overline{\Omega}$ in $H^2(M_I, \mathcal{O})$ is non-trivial** (can be false when the canonical bundle is non-trivial, or when (M, I, J, K) admits no HKT-structures). **Its top power is non-trivial in $H^{2n}(M_I, \mathcal{O})$** (L^k acts as an isomorphism from $H^{n-k}(M_I, \mathcal{O})$ to $H^{n+k}(M_I, \mathcal{O})$).

Hypercomplex structures on Kähler manifolds

0. Let (M, I) be a manifold admitting Kähler structure and a hypercomplex structure. **The canonical class of (M, I) is trivial by Calabi-Yau.**

1. From a Kähler form, **an HKT form is obtained by averaging with $SU(2)$** (the group of unitary quaternions, acting on TM).

2. **The cohomology class of Ω is non-trivial** by HKT-Lefschetz.

3. Since (M, I) is Kähler, this class is represented by a holomorphic form $\tilde{\Omega}$. **The top power of this class is non-trivial**, by HKT-Lefschetz.

4. The top power of $\tilde{\Omega}^n$ is a non-trivial holomorphic section of the canonical class, which is trivial. Therefore, $\tilde{\Omega}^n$ is nowhere vanishing, and **(M, I) is holomorphically symplectic.**

5. **Use Calabi-Yau to obtain that it is hyperkähler.**

Exotic hypercomplex structures

REMARK: The main theorem implies that a compact complex manifold which admits an Kähler structure and a hypercomplex structure, also admits a hyperkähler structure. However, **this does not rule out the following scenario.**

REMARK: A hypercomplex manifold (M, I, J, K) **is hyperkähler of and only if its Obata holonomy group is compact**.

DEFINITION: Let (M, I) be a compact, complex, holomorphically symplectic manifold of Kähler type, and (M, I, J, K) a hypercomplex structure. It is called **exotic** if its Obata holonomy is not compact.

QUESTION: Do exotic hypercomplex structures exist?

CONJECTURE: No, they don't.

REMARK: Should be possible to prove this exlicitly for a torus; it would be a good master diploma work. In real dimension 4, classification of hypercomplex structures is due to Boyer, and **it also implies the non-existence of exotic hypercomplex structures.** Maybe his argument can be generalized?

Lefschetz identities for general HKT-manifolds.

A full strength theorem (we don't need it, in fact it was never used AFAIK).

THEOREM: Let (M, I, J, K, g) be a hypercomplex manifold, K the canonical bundle of $M_I := (M, I)$, $K^{1/2}$ its square root (considered as a holomorphic vector bundle). Consider the map

$$L: \ H^i(M_I, K^{1/2}) \longrightarrow H^{i+2}(M_I, K^{1/2})$$

mapping a class represented by a form

$$\eta \in \Lambda^{0,p}(M_I) \otimes K^{1/2}$$

to $\eta \wedge \overline{\Omega}$ (this defines a correct operation on cohomology, because $\overline{\Omega}$ is $\overline{\partial}$-closed). **Then L is an element in an $\mathfrak{sl}(2)$-triple acting on $H^i(M_I, K^{1/2})$.**

It is a theorem about harmonic spinors. When $K^{1/2}$ is non-trivial, the cohomology groups $H^i(M_I, K^{1/2})$ are often empty.