Kähler geometry

lecture 2

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REMINDER: Complex structure on vector spaces

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. Such an automorphism is called **a complex structure operator** on V.

DEFINITION: The vector space over $\mathbb{C}$ with the same basis is called **a complexification** of V, denoted $V \otimes_{\mathbb{R}} \mathbb{C}$.

CLAIM: For an appropriate basis in $V \otimes_{\mathbb{R}} \mathbb{C}$, the complex structure operatorcan be written as

$$I = \begin{pmatrix} \sqrt{-1} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \sqrt{-1} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -\sqrt{-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & -\sqrt{-1} \end{pmatrix},$$

with the eigenspaces of equal dimension.

REMINDER: Hermitian structures

DEFINITION: Let (V, I) be a real vector space with a complex structure. A scalar product is called **I-invariant**, if $g(Ix, Iy) = g(x, y)$. An I-invariant positive definite scalar product on (V, I) is called **an Hermitian metric on V**, and (V, I, g) – an Hermitian space.

REMARK: Let I be a complex structure operator on a real vector space V, and g – a Hermitian metric. Then **the bilinear form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.** Indeed, $\omega(x, y) = g(x, Iy) = g(Ix, I^2 y) = -g(Ix, y) = -\omega(y, x)$.

DEFINITION: A skew-symmetric form $\omega(x, y)$ is called **an Hermitian form on (V, I)**.

REMARK: In the triple I, g, ω, each element can recovered from the other two.

REMINDER: tensor product

DEFINITION: Let V, W be vector spaces over a field $k = \mathbb{R}$ or $\mathbb{C}$, and V^*, W^* the dual spaces. Denote by $\operatorname{Bil}(V \times W, k)$ the space of bilinear maps from V, W to k, and let $V \otimes W$ denote $\operatorname{Bil}(V^* \times W^*, k)$, where V^*, W^* are dual spaces to V, W. The space $V \otimes W$ is called **the tensor product** of V, W.

DEFINITION: Given $v, w \in V, W$, one has an element $v \otimes w \in \operatorname{Bil}(V^* \times W^*, k)$, mapping a pair of functionals $\lambda \in V^*, \mu \in W^*$ to $\lambda(v)\mu(w)$. This vector is called **the tensor product** of v and w.

CLAIM: If $\{v_i, i = 1, ..., n\}$ is a basis in V, $\{w_j, j = 1, ..., m\}$ a basis in W, **the vectors $\{v_i \otimes w_j, i = 1, ..., n, j = 1, ..., m\}$ give a basis in $V \otimes W$.** In particuler, **$V \otimes W$ is $(\dim V \cdot \dim W)$-dimensional.**

REMARK: The tensor product is uniquely defined by the following **universal property**. Each bilinear map $B : (V, W) \longrightarrow k$ can be extended uniquely to the map $B^{\otimes} : V \otimes W \longrightarrow k$, in such a way that $B^{\otimes}(v \otimes w) = B(v, w)$.

EXERCISE: Prove this.

REMINDER: tensor algebra

DEFINITION: Given several vector spaces $V_1, ..., V_n$, their **tensor product** $V_1 \otimes V_2 \otimes ... \otimes V_n$ is defined as $V_1 \otimes (V_2 \otimes (V_2 \otimes ...V_n))...)$.

CLAIM: The tensor product operation **is commutative and associative.** Moreover, **the space $V_1 \otimes V_2 \otimes ... \otimes V_n$ is isomorphic to the space $B(V_1^*, V_2^*, ..., V_n^*)$** of polylinear maps from $V_1^*, V_2^*, ..., V_n^*$ to k.

EXERCISE: Prove this.

REMARK: Given vectors in $v \in V_1 \otimes V_2 \otimes ... \otimes V_n$ and $w \in W_1 \otimes W_2 \otimes ... \otimes W_m$, the tensor product $v \otimes w$ sits in $V_1 \otimes V_2 \otimes ... \otimes V_n \otimes W_1 \otimes W_2 \otimes ... \otimes W_m$.

CLAIM: This defines the structure of an algebra on $T^{\otimes} V = k \oplus V \oplus V \otimes V \oplus ... \oplus V^{\otimes n}$, where $V^{\otimes n}$ is a tensor product of n copies of V.

EXERCISE: Prove this.

DEFINITION: The algebra $T^{\otimes} V$ is called **the tensor algebra**, or **free algebra** generated by V.

REMINDER: The Grassmann algebra

EXERCISE: Prove that **any algebra generated by V can be obtained as a quotient of $T^{\otimes}V$ by an ideal.**

EXERCISE: Let $v_1, ..., v_n \in V$ be a basis. Prove that **the polynomial algebra $k[v_1, ..., v_n]$ is a quotient of $T^{\otimes}V$ by an ideal generated by $x \otimes y - y \otimes x$, for all $x, y \in V$.**

DEFINITION: A Grassmann algebra $\Lambda^* V$ is a quotient of $T^{\otimes}V$ by an ideal generated by $x \otimes y + y \otimes x$, for all $x, y \in V$. The multiplication in $\Lambda^* V$ is denoted by $x, y \longrightarrow x \wedge y$, called **the wedge product**.

Properties of Grassmann algebra:

1. $\dim \Lambda^i V := \binom{\dim V}{i}$, $\dim \Lambda^* V = 2^{\dim V}$.

2. $\Lambda^*(V \oplus W) = \Lambda^*(V) \otimes \Lambda^*(W)$.

REMINDER: Vector fields

DEFINITION: Let X be the vector field on a manifold M, and f a function. Denote by $\operatorname{Lie}_X f$ **the derivatiive** of f along X.

DEFINITION: **A derivation** on a commutative ring is a map $R \xrightarrow{d} R$ satisfying **the Leibniz identity** $d(xy) = d(x)y + xd(y)$.

THEOREM: Each derivation of the ring $C^\infty M$ of smooth functions on M is given by a vector field X; **this correspondence is bijective.**

REMARK: This can be used as a definition of a vector field.

EXERCISE: Prove that **a commutator of two derivations is again a derivation.**

REMARK: Vector fields are the same as derivations of $C^\infty M$. This allows us to define **the commutator of two vector fields** as the commutator of the corresponding derivations.

DEFINITION: Denote by TM the bundle of vector fields, and by $\Lambda^1 M$ or T^* the dual bundle, called **the bundle of 1-forms**. For any $f \in C^\infty M$, the operation $X \longrightarrow \operatorname{Lie}_X f$ is linear as a function of X, hence it defines a section of T^*M. We denote this section df, and call it **the differential** of f.

REMINDER: de Rham algebra

DEFINITION: Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T_x^* M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassman algebra of the cotangent space $T_x^* M$). The sections of $\Lambda^i M$ are called **differential i-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

REMARK: $\Lambda^0 M = C^\infty M$.

THEOREM: **There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} \ldots$ satisfying the following properties**

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

EXERCISE: **Prove it**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

REMINDER: Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

REMARK: The commutator defines a $\mathbb{C}^\infty M$-linear map $N := \Lambda^2(T^{1,0}) \longrightarrow T^{0,1}M$, called **the Nijenhuis tensor** of I. **One can represent N as a section of $\Lambda^{2,0}(M) \otimes T^{0,1}M$.**

EXAMPLE: Symmetric spaces.

EXAMPLE: $\mathbb{C}P^n$.

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^2(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$ using the Haar measure on $U(n+1)$.

EXERCISE: Prove that **the Fubini-Study form is unique** (up to a constant multiplier).

Examples of Kähler manifolds.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n = \mathbb{C}^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: Fubini-Study form is Kähler. Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish, because $-\operatorname{Id} \in U(n)$

REMARK: The same argument works for all symmetric spaces.

DEFINITION: An almost complex submanifold $X \subset M$ of an almost complex manifold (M, I) is a smooth submanifold which satisfies $I(TX) \subset TX$.

EXERCISE: Let $X \subset M$ be an almost complex submanifold of (M, I), where I is integrable. **Prove that $(X, I|_{TX})$ is a complex manifold.**

DEFINITION: In this situation, X is called **a complex submanifold** of M.

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler. Indeed, a restriction of a closed form is again closed.

Connections

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: A **connection** on a vector bundle B is a map $B \xrightarrow{\nabla} \Lambda^1 M \otimes B$ which satisfies

$$\nabla(fb) = df \otimes b + f\nabla b$$

for all $b \in B$, $f \in C^\infty M$.

REMARK: A connection ∇ on B gives a connection $B^* \xrightarrow{\nabla^*} \Lambda^1 M \otimes B^*$ on the dual bundle, by the formula

$$d(\langle b, \beta\rangle) = \langle \nabla b, \beta\rangle + \langle b, \nabla^* \beta\rangle$$

These connections are usually denoted **by the same letter ∇.**

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

Torsion

DEFINITION: **The torsion** of a connection $\Lambda^1 \stackrel{\nabla}{\longrightarrow} \Lambda^1 M \otimes \Lambda^1 M$ is a map $\operatorname{Alt} \circ \nabla - d$, where $\operatorname{Alt} : \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication. It is a map $T_\nabla : \Lambda^1 M \longrightarrow \Lambda^2 M$.

EXERCISE: **Prove that torsion is a $C^\infty M$-linear.**

REMARK: The dual operator $x, y \longrightarrow \nabla_x Y - \nabla_y X - [X, Y]$ is also called **the torsion of ∇**. It is a map $\Lambda^2 TM \longrightarrow TM$.

EXERCISE: **Prove that these two tensors are dual.**

DEFINITION: Let (M, g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g) = 0$. It is called **Levi-Civita** if it is torsion-free.

THEOREM: ("the main theorem of differential geometry")
For any Riemannian manifold, the Levi-Civita connection exists, and it is unique.

Levi-Civita connection and Kähler geometry

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) **The complex structure I is integrable, and the Hermitian form ω is closed.**

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection.

REMARK: The implication (ii) $\Rightarrow$ (i) is clear. Indeed, $[X, Y] = \nabla_X Y - \nabla_Y X$, hence it is a $(1,0)$-vector field when X, Y are of type (1,0), and then I **is integrable**. Also, $d\omega = 0$, **because ∇ is torsion-free,** and $d\omega = \operatorname{Alt}(\nabla\omega)$.

The implication (i) $\Rightarrow$ (ii) is proven by the same argument as used to construct the Levi-Civita connection.

Holonomy group

DEFINITION: (Cartan, 1923) Let (B, ∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B, ∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: A bundle is **flat** (has vanishing curvature) **if and only if its restricted holonomy vanishes.**

REMARK: If $\nabla(\varphi) = 0$ for some tensor $\varphi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$, **the holonomy group preserves φ.**

DEFINITION: **Holonomy of a Riemannian manifold** is holonomy of its Levi-Civita connection.

EXAMPLE: Holonomy of a Riemannian manifold lies in $O(T_xM, g|_x) = O(n)$.

EXAMPLE: Holonomy of a Kähler manifold lies in $U(T_xM, g|_x, I|_x) = U(n)$.

REMARK: The holonomy group **does not depend on the choice of a point $x \in M$.**

Curvature of a connection

Let M be a manifold, B a bundle, $\Lambda^i M$ the differential forms, and $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ a connection. We extend ∇ to $B \otimes \Lambda^i M \xrightarrow{\nabla} B \otimes \Lambda^{i+1} M$ in a natural way, using the formula

$$\nabla(b \otimes \eta) = \nabla(b) \wedge \eta + b \otimes d\eta,$$

and define **the curvature Θ_∇** of ∇ as $\nabla \circ \nabla : B \longrightarrow B \otimes \Lambda^2 M$.

CLAIM: This operator is $C^\infty M$-linear.

REMARK: We shall consider Θ_∇ as an element of $\Lambda^2 M \otimes \operatorname{End} B$, that is, an $\operatorname{End} B$-valued 2-form.

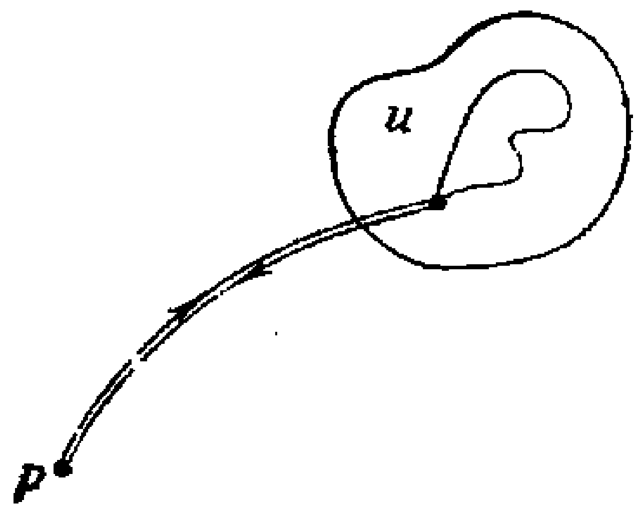
REMARK: Given vector fields $X, Y \in TM$, the curvature can be written in terms of a connection as follows

$$\Theta_\nabla(b) = \nabla_X \nabla_Y b - \nabla_Y \nabla_X B - \nabla_{[X,Y]} b.$$

CLAIM: Suppose that the structure group of B is reduced to its subgroup G, and let ∇ be a connection which preserves this reduction. This is the same as to say that the connection form takes values in $\Lambda^1 \otimes \mathfrak{g}(B)$. **Then Θ_∇ lies in $\Lambda^2 M \otimes \mathfrak{g}(B)$.**

The Lasso lemma

DEFINITION: A **lasso** is a loop of the following form:



The round part is called **a working part** of a loop.

REMARK: ("The Lasso Lemma") Let $\{U_i\}$ be a covering of a manifold, and γ a loop. Then **any contractible loop γ is a product of several lasso, with working part of each inside some U_i.**

The Ambrose-Singer theorem

DEFINITION: Let (B,∇) be a bundle with connection, $\Theta \in \Lambda^2(M)\otimes \operatorname{End}(B)$ its curvature, and $a, b \in T_xM$ tangent vectors. An endomorphism $\Theta(a,b) \in \operatorname{End}(B)|_x$ is called **a curvature element**.

THEOREM: (Ambrose-Singer) The restricted holonomy group of B, ∇ at $z \in M$ is a Lie group, **with its Lie algebra generated by all curvature elements $\Theta(a,b) \in \operatorname{End}(B)|_x$ transported to z along all paths.**

REMARK: Its proof follows from the Lasso lemma.

Holonomy representation

DEFINITION: Let (M, g) be a Riemannian manifold, G its holonomy group. A **holonomy representation** is the natural action of G on TM.

THEOREM: (de Rham) Suppose that the holonomy representation is not irreducible: $T_xM = V_1 \oplus V_2$. Then M locally splits as $M = M_1 \times M_2$, with $V_1 = TM_1$, $V_2 = TM_2$.

Proof. Step 1: Using the parallel transform, we extend $V_1 \oplus V_2$ to a **splitting of vector bundles $TM = B_1 \oplus B_2$, preserved by holonomy.**

Step 2: The sub-bundles B_1, $B_2 \subset TM$ **are integrable:** $[B_1, B_1] \subset B_i$ (the Levi-Civita connection is torsion-free)

Step 3: Taking the leaves of these integrable distributions, **we obtain a local decomposition $M = M_1 \times M_2$, with $V_1 = TM_1$, $V_2 = TM_2$.**

Step 4: Since the splitting $TM = B_1 \oplus B_2$ is preserved by the connection, **the leaves M_1, M_2 are totally geodesic.**

Step 5: Therefore, **locally M splits (as a Riemannian manifold)**: $M = M_1 \times M_2$, where M_1, M_2 are any leaves of these foliations. ■

The de Rham splitting theorem

COROLLARY: Let M be a Riemannian manifold, and $\mathcal{H}ol_0(M) \xrightarrow{\rho} \operatorname{End}(T_xM)$ a reduced holonomy representation. Suppose that ρ is reducible: $T_xM = V_1 \oplus V_2 \oplus ... \oplus V_k$. **Then $G = \mathcal{H}ol_0(M)$ also splits: $G = G_1 \times G_2 \times ... \times G_k$,** with each G_i acting trivially on all V_j with $j \neq i$.

Proof: Locally, this statement follows from the local splitting of M proven above. To obtain it globally in M, use the Lasso Lemma. ■

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

REMARK: It is easy to find non-complete or non-simply connected counterexamples to de Rham theorem.

THEOREM: (Simons, 1962) Let M be a manifold with irreducible holonomy. **Then either M is locally symmetric, or $\mathcal{H}ol(M)$ acts transitively on the unit sphere in T_xM.**

Berger's theorem

THEOREM: (Berger's theorem, 1955) Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

REMARK: There is one more group acting transitively on a sphere: $Spin(9)$ acting on $S^{15} \subset \mathbb{R}^{16}$. In 1968, D. Alekseevsky has shown that **a manifold with holonomy $Spin(9)$ is automatically locally symmetric.**

REMARK: A similar list exists for non-orthogonal irreducible holonomy without torsion (Merkulov, Schwachhöfer, 1999).