Kähler geometry

lecture 3

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REMINDER: Complex structure on vector spaces

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. Such an automorphism is called **a complex structure operator** on V.

DEFINITION: The vector space over $\mathbb{C}$ with the same basis is called **a complexification** of V, denoted $V \otimes_{\mathbb{R}} \mathbb{C}$.

CLAIM: For an appropriate basis in $V \otimes_{\mathbb{R}} \mathbb{C}$, the complex structure operatorcan be written as

$$I = \begin{pmatrix} \sqrt{-1} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \sqrt{-1} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -\sqrt{-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & -\sqrt{-1} \end{pmatrix},$$

with the eigenspaces of equal dimension.

REMINDER: Hermitian structures

DEFINITION: Let (V, I) be a real vector space with a complex structure. A scalar product is called **I-invariant**, if $g(Ix, Iy) = g(x, y)$. An I-invariant positive definite scalar product on (V, I) is called **an Hermitian metric on V**, and (V, I, g) – an Hermitian space.

REMARK: Let I be a complex structure operator on a real vector space V, and g – a Hermitian metric. Then **the bilinear form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.** Indeed, $\omega(x, y) = g(x, Iy) = g(Ix, I^2y) = -g(Ix, y) = -\omega(y, x)$.

DEFINITION: A skew-symmetric form $\omega(x, y)$ is called **an Hermitian form on (V, I)**.

REMARK: In the triple I, g, ω, each element can recovered from the other two.

REMINDER: Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

REMARK: The commutator defines a $\mathbb{C}^\infty M$-linear map $N := \Lambda^2(T^{1,0}) \longrightarrow T^{0,1}M$, called **the Nijenhuis tensor** of I. **One can represent N as a section of $\Lambda^{2,0}(M) \otimes T^{0,1}M$.**

EXAMPLE: Symmetric spaces.

EXAMPLE: $\mathbb{C}P^n$.

REMINDER: Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^2(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

REMINDER: Projective manifolds.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$ using the Haar measure on $U(n+1)$.

Claim: Fubini-Study form is Kähler. Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish, because $-\operatorname{Id} \in U(n)$

REMARK: The same argument works for all symmetric spaces.

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler. Indeed, a restriction of a closed form is again closed.

REMINDER: Connections

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: A **connection** on a vector bundle B is a map $B \xrightarrow{\nabla} \Lambda^1 M \otimes B$ which satisfies

$$\nabla(fb) = df \otimes b + f\nabla b$$

for all $b \in B$, $f \in C^\infty M$.

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula.

REMINDER: Levi-Civita connection

DEFINITION: Let (M,g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g) = 0$. It is called **Levi-Civita** if it is torsion-free.

THEOREM: ("the main theorem of differential geometry") **For any Riemannian manifold, the Levi-Civita connection exists, and it is unique**.

THEOREM: Let (M,I,g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) **The complex structure I is integrable, and the Hermitian form ω is closed.**

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection.

REMINDER: Ambrose-Singer theorem

DEFINITION: (Cartan, 1923) Let (B, ∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla} : B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B, ∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

DEFINITION: Let (B, ∇) be a bundle with connection, $\Theta \in \Lambda^2(M) \otimes \operatorname{End}(B)$ its curvature, and $a, b \in T_x M$ tangent vectors. An endomorphism $\Theta(a, b) \in \operatorname{End}(B)|_x$ is called **a curvature element**.

THEOREM: (Ambrose-Singer) The restricted holonomy group of B, ∇ at $z \in M$ is a Lie group, **with its Lie algebra generated by all curvature elements $\Theta(a, b) \in \operatorname{End}(B)|_x$ transported to z along all paths.**

REMINDER: Holonomy representation

DEFINITION: Let (M,g) be a Riemannian manifold, G its holonomy group. A **holonomy representation** is the natural action of G on TM.

THEOREM: (de Rham) Suppose that the holonomy representation is not irreducible: $T_xM = V_1 \oplus V_2$. Then M locally splits as $M = M_1 \times M_2$, with $V_1 = TM_1$, $V_2 = TM_2$.

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

REMARK: It is easy to find non-complete or non-simply connected counterexamples to de Rham theorem.

THEOREM: (Simons, 1962) Let M be a manifold with irreducible holonomy. **Then either M is locally symmetric, or $\mathcal{H}\mathrm{ol}(M)$ acts transitively on the unit sphere in T_xM.**

Berger's theorem

THEOREM: (Berger's theorem, 1955) Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

REMARK: There is one more group acting transitively on a sphere: $Spin(9)$ acting on $S^{15} \subset \mathbb{R}^{16}$. In 1968, D. Alekseevsky has shown that **a manifold with holonomy $Spin(9)$ is automatically locally symmetric.**

REMARK: A similar list exists for non-orthogonal irreducible holonomy without torsion (Merkulov, Schwachhöfer, 1999).

Holomorphic vector bundles

DEFINITION: A (smooth) **vector bundle** on a smooth manifold is a locally trivial sheaf of $C^\infty M$-modules.

DEFINITION: A **holomorphic vector bundle** on a complex manifold is a locally trivial sheaf of $\mathcal{O}_M$-modules.

REMARK: A section b of a bundle B is often denoted as $b \in B$.

CLAIM: Let B be a holomorphic vector bundle. Consider the sheaf $B_{C^\infty} := B \otimes_{\mathcal{O}_M} C^\infty M$. It is clearly locally trivial, hence **B_{C^∞} is a smooth vector bundle.**

DEFINITION: B_{C^∞} is called **a smooth vector bundle underlying B.**

A holomorphic structure operator

DEFINITION: Let $d = d^{0,1} + d^{1,0}$ be the Hodge decomposition of the de Rham differential on a complex manifold, $d^{0,1}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M)$ and $d^{1,0}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M)$. The operators $d^{0,1}$, $d^{1,0}$ are denoted $\overline{\partial}$ and ∂ and called **the Dolbeault differentials**.

REMARK: From $d^2 = 0$, one obtains $\overline{\partial}^2 = 0$ **and** $\partial^2 = 0$**.**

REMARK: The operator $\overline{\partial}$ is $\mathcal{O}_M$-linear.

DEFINITION: Let B be a holomorphic vector bundle, and $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ an operator mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where $b \in B$ is a holomorphic section, and f a smooth function. This operator is called **a holomorphic structure operator** on B. **It is correctly defined, because $\overline{\partial}$ is $\mathcal{O}_M$-linear.**

REMARK: The kernel of $\overline{\partial}$ **coincides with the set of holomorphic sections** of B.

The $\overline{\partial}$-operator on vector bundles

DEFINITION: A $\overline{\partial}$-operator on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator on B can be extended to

$$\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

REMARK: If $\overline{\partial}$ is a holomorphic structure operator, then $\overline{\partial}^2 = 0$.

THEOREM: (Atiyah-Bott) Let $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\overline{\partial}$-operator, satisfying $\overline{\partial}^2 = 0$. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic vector bundle of the same rank.**

REMARK: This statement is a vector bundle analogue of Newlander-Nirenberg theorem.

DEFINITION: $\overline{\partial}$-operator $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ on a smooth manifold is called **a holomorphic structure operator**, if $\overline{\partial}^2 = 0$.

Connections and holomorphic structure operators

DEFINITION: let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\, B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}: V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0}: V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: **An Hermitian holomorphic vector bundle** is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure.

DEFINITION: **A Chern connection** on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

Curvature of a connection

DEFINITION: Let $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a smooth budnle. Extend it to an operator on B-valued forms

$$B \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} ...$$

using $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde\eta}\eta \wedge \nabla b$. The operator $\nabla^2 : B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: The algebra of $\operatorname{End}(B)$-valued forms naturally acts on $\Lambda^* M \otimes B$. The curvature satisfies $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b = f\nabla^2 b$, hence it is $C^\infty M$-linear. **We consider it as an $\operatorname{End}(B)$-valued 2-form on M.**

PROPOSITION: (Bianchi identity) Using the **graded Jacobi identity**, we obtain $[\nabla, \nabla^2] = [\nabla^2, \nabla] + [\nabla, \nabla^2] = 0$, hence $[\nabla, \nabla^2] = 0$. This gives **Bianchi identity:** $\nabla(\Theta_B) = 0$.

REMARK: If B is a line bundle, $\operatorname{End} B$ is trivial, and **the curvature Θ_B of B is a closed 2-form.**

DEFINITION: The cohomology class $c_1(B) := \frac{\sqrt{-1}}{2\pi}[\Theta_B] \in H^2(M)$ is called **the real first Chern class of a line bunlde B**.

An exercise: Check that $c_1(B)$ is independent from a choice of ∇.

Curvature of a holomorphic line bundle

REMARK: When speaking of a **"curvature of a holomorphic bundle"**, one usually means the curvature of a Chern connection.

REMARK: Let B be a holomorphic Hermitian line bundle, and b its non-degenerate section. Denote by η a (1,0)-form which satisfies $\nabla^{1,0}b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0}b, b) = \operatorname{Re} \eta |b|^2$. **This gives** $\nabla^{1,0}b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b.$

REMARK: Then $\Theta_B(b) = 2\overline{\partial}\partial \log |b| b$, **that is,** $\Theta_B = -2\partial\overline{\partial} \log |b|$.

COROLLARY: If $g' = e^{2f}g$ – two metrics on a holomorphic line bundle, Θ, Θ' their curvatures, **one has** $\Theta' - \Theta = -2\partial\overline{\partial} f$

CLAIM: Let η be a closed (1,1)-form in the same cohomology class as $\Theta_{B,h}$. **Then η is a curvature of a Chern connection** on B, for some metric h'.

Proof: The difference $\Theta_{B,h} - \eta$ is an exact (1,1)-form, hence **belongs to an image of $\partial\overline{\partial}$ ("$\partial\overline{\partial}$-lemma"): $\Theta_{B,h} - \eta = -2\partial\overline{\partial} f$.** Then the curvature of a metric $h' := e^{2f}h$ satisfies $\Theta_{B,h} - \Theta_{B,h'} = -2\partial\overline{\partial} f$, hence $\eta = \Theta_{B,h'}$. ■

REMARK: Such metric is unique, up to a constant.

Calabi-Yau manifolds

REMARK: Let B be a line bundle on a manifold. Using the long exact sequence of cohomology associated with the exponential sequence

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow C^\infty M \longrightarrow (C^\infty M)^* \longrightarrow 0,$$

we obtain $0 \longrightarrow H^1(M, (C^\infty M)^*) \longrightarrow H^2(M, \mathbb{Z}) \longrightarrow 0$**.**

DEFINITION: Let B be a complex line bundle, and ξ_B its defining element in $H^1(M, (C^\infty M)^*)$. Its image in $H^2(M, \mathbb{Z})$ is called **the integer first Chern class** of B.

REMARK: A complex line bundle B is (topologically) trivial if and only if $c_1(B) = 0$.

THEOREM: (Gauss-Bonnet) A real Chern class of a vector bundle **is an image of the integer Chern class $c_1(B, \mathbb{Z})$** under the natural homomorphism $H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{R})$.

DEFINITION: **A first Chern class** of a complex n-manifold is $c_1(\Lambda^{n,0}(M))$.

DEFINITION:
A Calabi-Yau manifold is a compact Kaehler manifold with $c_1(M, \mathbb{Z}) = 0$.

Calabi-Yau theorem

DEFINITION: Let (M, I, ω) be a Kaehler n-manifold, and $K(M) := \Lambda^{n,0}(M)$ its **canonical bundle.** We consider $K(M)$ as a holomorphic line bundle, $K(M) = \Omega^n M$. The natural Hermitian metric on $K(M)$ is written as

$$(\alpha, \alpha') \longrightarrow \frac{\alpha \wedge \overline{\alpha}'}{\omega^n}.$$

Denote by Θ_K the curvature of the Chern connection on $K(M)$. The **Ricci curvature** Ric of M is symmetric 2-form $\operatorname{Ric}(x, y) = \Theta_K(x, Iy)$.

DEFINITION: A Kähler manifold is called **Ricci-flat** if its Ricci curvature vanishes.

THEOREM: (Calabi-Yau)
Let (M, I, g) be Calabi-Yau manifold. **Then there exists a unique Ricci-flat Kaehler metric in any given Kaehler class.**

Calabi-Yau theorem and Monge-Ampère equation

REMARK: Let (M,ω) be a Kähler n-fold, and Ω a non-degenerate section of $K(M)$, Then $|\Omega|^2 = \frac{\Omega\wedge\overline{\Omega}}{\omega^n}$ If ω_1 is a new Kaehler metric on (M,I), h, h_1 the associated metrics on $K(M)$, then $\frac{h}{h_1} = \frac{\omega_1^n}{\omega^n}$

COROLLARY: A metric $\omega_1 = \omega + \partial\overline{\partial}\varphi$ **is Ricci-flat if and only if** $(\omega + \partial\overline{\partial}\varphi)^n = \omega^n e^f$**,** where $-2\partial\overline{\partial} f = \Theta_{K,\omega}$.

Proof: For such f, φ, one has $\log\frac{h}{h_1} = -f$. This gives

$$\Theta_{K,\omega_1} = \Theta_{K,\omega} + \partial\overline{\partial}\frac{h}{h_1} = \Theta_{K,\omega} - 2\partial\overline{\partial} f = 0.$$

■

THEOREM: (Calabi-Yau) Let (M,ω) be a compact Kaehler n-manifold, and f any smooth function. **Then there exists a unique up to a constant function** φ such that $(\omega + dd^c\varphi)^n = Ae^f\omega^n$, where A is a positive constant obtained from the formula $\int_M Ae^f\omega^n = \int_M \omega^n$.

REMARK:

$$(\omega + dd^c\varphi)^n = Ae^f\omega^n,$$

is called **the Monge-Ampere equation.**

Uniqueness of solutions of complex Monge-Ampere equation

PROPOSITION: (Calabi) **A complex Monge-Ampere equation has at most one solution,** up to a constant.

Proof. Step 1: Let ω_1, ω_2 be solutions of Monge-Ampere equation. Then $\omega_1^n = \omega_2^n$. By dd^c-lemma, one has $\omega_2 = \omega_1 + dd^c\psi$. **We need to show $\psi = const$.**

Step 2: This gives

$$0 = (\omega_1 + dd^c\psi)^n - \omega_1^n = dd^c\psi \wedge \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}.$$

Step 3: Let $P := \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}$. This is a positive $(n-1, n-1)$-form. **There exists a Hermitian form ω_3 on M such that $\omega_3^{n-1} = P$.**

Step 4: Since $dd^c\psi \wedge P = 0$, this gives $\psi dd^c\psi \wedge P = 0$. Stokes' formula implies

$$0 = \int_M \psi \wedge \partial\bar\partial\psi \wedge P = -\int_M \partial\psi \wedge \bar\partial\psi \wedge P = -\int_M |\partial\psi|_3^2 \omega_3^n.$$

where $|\cdot|_3$ is the metric associated to ω_3. **Therefore $\bar\partial\psi = 0$.** ■