Kähler geometry

lecture 4

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REMINDER: The Grassmann algebra

REMARK: Given vectors in $v \in V_1 \otimes V_2 \otimes ... \otimes V_n$ and $w \in W_1 \otimes W_2 \otimes ... \otimes W_m$, the tensor product $v \otimes w$ sits in $V_1 \otimes V_2 \otimes ... \otimes V_n \otimes W_1 \otimes W_2 \otimes ... \otimes W_m$.

CLAIM: This defines the structure of an algebra on $T^{\otimes} V = k \oplus V \oplus V \otimes V \oplus ... \oplus V^{\otimes n}$, where $V^{\otimes n}$ is a tensor product of n copies of V.

DEFINITION: The algebra $T^{\otimes} V$ is called **the tensor algebra**, or **free algebra** generated by V.

EXERCISE: Prove that **any algebra generated by V can be obtained as a quotient of $T^{\otimes} V$ by an ideal.**

DEFINITION: A Grassmann algebra $\Lambda^* V$ is a quotient of $T^{\otimes} V$ by an ideal generated by $x \otimes y + y \otimes x$, for all $x, y \in V$. The multiplication in $\Lambda^* V$ is denoted by $x, y \longrightarrow x \wedge y$, called **the wedge product**.

Properties of Grassmann algebra:

1. $\dim \Lambda^i V := \binom{\dim V}{i}$, $\dim \Lambda^* V = 2^{\dim V}$.

2. $\Lambda^*(V \oplus W) = \Lambda^*(V) \otimes \Lambda^*(W)$.

REMINDER: de Rham algebra

DEFINITION: Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T_x^* M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassman algebra of the cotangent space $T_x^* M$). The sections of $\Lambda^i M$ are called **differential i-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

REMARK: $\Lambda^0 M = C^\infty M$.

THEOREM: **There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} ...$ satisfying the following properties**

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

EXERCISE: **Prove it**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

REMINDER: The Hodge decomposition in linear algebra

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

CLAIM: $\Lambda^*(V \oplus W) = \Lambda^*(V) \otimes \Lambda^*(W)$

REMARK: Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ induces $\Lambda^*_{\mathbb{C}}(V) = \Lambda^*_{\mathbb{C}}(V^{0,1}) \otimes \Lambda^*_{\mathbb{C}}(V^{1,0})$, giving

$$\Lambda^d V_{\mathbb{C}} = \bigoplus_{p+q=d} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}.$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

REMINDER: Cauchy-Riemann equation and Hodge decomposition

The (p,q)-decomposition is defined on differential forms on complex manifold, in a similar way.

DEFINITION: Let (M,I) be a complex manifold A differential form $\eta \in \Lambda^1(M)$ is **of type (1,0)** if $I(\eta) = \sqrt{-1}\,\eta$, and **of type (0,1)** if $I(\eta) = -\sqrt{-1}\,\eta$. The corresponding vector bundles are denoted by $\Lambda^{1,0}(M)$ and $\Lambda^{0,1}(M)$.

REMARK: Cauchy-Riemann equations can be written as $df \in \Lambda^{1,0}(M)$. That is, **a function $f \in C^\infty_{\mathbb{C}}(M)$ is holomorphic if and only if $df \in \Lambda^{1,0}(M)$.**

REMARK: Let (M,I) be a complex manifold, and $z_1, ..., z_n$ holomorphic coordinate system in $U \subset M$, with z_i being holomorphic functions on U. **Then $dz_1, ..., dz_n$ generate the bundle $\Lambda^{1,0}(M)$, and $d\bar{z}_1, ..., d\bar{z}_n$ generate $\Lambda^{0,1}(M)$.**

EXERCISE: Prove this.

REMINDER: The Hodge decomposition on a complex manifold

DEFINITION: Let (M,I) be a complex manifold, $\{U_i\}$ its covering, and and $z_1, ..., z_n$ holomorphic coordinate system on each covering patch. **The bundle $\Lambda^{p,q}(M,I)$ of (p,q)-forms on (M,I)** is generated locally on each coordinate patch by monomials $dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} \wedge d\overline{z}_{i_{p+1}} \wedge ... \wedge dz_{i_{p+q}}$. **The Hodge decomposition** is a decomposition of vector bundles:

$$\Lambda^d_{\mathbb{C}}(M) = \bigoplus_{p+q=d} \Lambda^{p,q}(M).$$

REMARK: One has $\Lambda^{p,q}(M) = \Lambda^{p,0}(M) \otimes \Lambda^{0,q}(M)$. This gives $\operatorname{rk} \Lambda^{p,q}(M) = \binom{n}{p} \cdot \binom{n}{q}$, where $n = \dim_{\mathbb{C}} M$.

EXERCISE: Prove that the **de Rham differential on a complex manifold has only two Hodge components:**

$$d\left(\Lambda^{p,q}(M)\right) \subset \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M).$$

DEFINITION: Let $d = d^{0,1} + d^{1,0}$ be the Hodge decomposition of the de Rham differential on a complex manifold, $d^{0,1}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M)$ and $d^{1,0}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M)$. The operators $d^{0,1}$, $d^{1,0}$ are denoted $\overline{\partial}$ and ∂ and called **the Dolbeault differentials**.

REMINDER: Holomorphic vector bundles

DEFINITION: A $\overline{\partial}$-operator on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator on B can be extended to

$$\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

DEFINITION: A holomorphic vector bundle on a complex manifold (M, I) is a vector bundle equipped with a $\overline{\partial}$-operator which satisfies $\overline{\partial}^2 = 0$. In this case, $\overline{\partial}$ is called **a holomorphic structure operator**.

EXERCISE: Consider the Dolbeault differential $\overline{\partial}: \Lambda^{p,0}(M) \longrightarrow \Lambda^{p,1}(M) = \Lambda^{p,0}(M) \otimes \Lambda^{0,1}(M)$. **Prove that it is a holomorphic structure operator on $\Lambda^{p,0}(M)$.**

DEFINITION: The corresponding holomorphic vector bundle $(\Lambda^{p,0}(M), \overline{\partial})$ is called **the bundle of holomorphic p-forms**, denoted by $\Omega^p(M)$.

REMINDER: Chern connection

DEFINITION: Let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\, B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1} : V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0} : V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: **An Hermitian holomorphic vector bundle** is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure operator $\overline{\partial}$.

DEFINITION: **A Chern connection** on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

REMINDER: Curvature of a connection

DEFINITION: Let $\nabla: B \longrightarrow B\otimes\Lambda^1 M$ be a connection on a smooth budnle. Extend it to an operator on B-valued forms

$$B \xrightarrow{\nabla} \Lambda^1(M)\otimes B \xrightarrow{\nabla} \Lambda^2(M)\otimes B \xrightarrow{\nabla} \Lambda^3(M)\otimes B \xrightarrow{\nabla} ...$$

using $\nabla(\eta\otimes b) = d\eta + (-1)^{\tilde\eta}\eta\wedge\nabla b$. The operator $\nabla^2: B \longrightarrow B\otimes\Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: The algebra of $\operatorname{End}(B)$-valued forms naturally acts on $\Lambda^* M\otimes B$. The curvature satisfies $\nabla^2(fb) = d^2 fb + df\wedge\nabla b - df\wedge\nabla b + f\nabla^2 b = f\nabla^2 b$, hence it is $C^\infty M$-linear. **We consider it as an $\operatorname{End}(B)$-valued 2-form on M.**

PROPOSITION: (Bianchi identity) Clearly, $[\nabla,\nabla^2] = [\nabla^2,\nabla] + [\nabla,\nabla^2] = 0$, hence $[\nabla,\nabla^2] = 0$. This gives **Bianchi identity:** $\nabla(\Theta_B) = 0$, where Θ is considered as a section of $\Lambda^2(M)\otimes\operatorname{End}(B)$, and $\nabla: \Lambda^2(M)\otimes\operatorname{End}(B) \longrightarrow \Lambda^3(M)\otimes\operatorname{End}(B)$. the operator defined above

REMINDER: Curvature of a holomorphic line bundle

REMARK: If B is a line bundle, $\operatorname{End} B$ is trivial, and **the curvature Θ_B of B is a closed 2-form.**

DEFINITION: Let ∇ be a unitary connection in a line bundle. The cohomology class $c_1(B) := \frac{\sqrt{-1}}{2\pi}[\Theta_B] \in H^2(M)$ is called **the real first Chern class of a line bunlde B**.

An exercise: Check that $c_1(B)$ is independent from a choice of ∇.

REMARK: When speaking of a **"curvature of a holomorphic bundle"**, one usually means the curvature of a Chern connection.

REMARK: Let B be a holomorphic Hermitian line bundle, and b its non-degenerate holomorphic section. Denote by η a (1,0)-form which satisfies $\nabla^{1,0}b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0}b, b) = \operatorname{Re} \eta |b|^2$. **This gives $\nabla^{1,0}b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b$.**

REMARK: Then $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, **that is, $\Theta_B = -2\partial\bar\partial \log |b|$.**

COROLLARY: If $g' = e^{2f}g$ – two metrics on a holomorphic line bundle, Θ, Θ' their curvatures, **one has $\Theta' - \Theta = -2\partial\bar\partial f$**

$\partial\bar\partial$-lemma

THEOREM: ("$\partial\bar\partial$-lemma")
Let M be a compact Kaehler manifold, and $\eta \Lambda^{p,q}(M)$ an exact form. Then $\eta = \partial\bar\partial\alpha$, for some $\alpha \in \Lambda^{p-1,q-1}(M)$.

Its proof uses Hodge theory.

COROLLARY: Let (L,h) be a holomorphic line bundle on a compact complex manifold, Θ its curvature, and η a (1,1)-form in the same cohomology class as $[\Theta]$. **Then there exists a Hermitian metric h' on L such that its curvature is equal to η.**

Proof: Let Θ' be the curvature of the Chern connection associated with h'. Then $\Theta' - \Theta = -2\partial\bar\partial f$, wgere $f = \log(h'h^{-1})$. Then $\Theta' - \Theta = \eta - \Theta = -2\partial\bar\partial f$ **has a solution f by $\partial\bar\partial$-lemma,** because $\eta - \Theta$ is exact. ■

Calabi-Yau manifolds

REMARK: Let B be a line bundle on a manifold. Using the long exact sequence of cohomology associated with the exponential sequence

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow C^\infty M \longrightarrow (C^\infty M)^* \longrightarrow 0,$$

we obtain $0 \longrightarrow H^1(M, (C^\infty M)^*) \longrightarrow H^2(M, \mathbb{Z}) \longrightarrow 0$**.**

DEFINITION: Let B be a complex line bundle, and ξ_B its defining element in $H^1(M, (C^\infty M)^*)$. Its image in $H^2(M, \mathbb{Z})$ is called **the integer first Chern class** of B, denoted by $c_1(B, \mathbb{Z})$ or $c_1(B)$.

REMARK: A complex line bundle B is (topologically) trivial if and only if $c_1(B, \mathbb{Z}) = 0$.

THEOREM: (Gauss-Bonnet) A real Chern class of a vector bundle **is an image of the integer Chern class $c_1(B, \mathbb{Z})$** under the natural homomorphism $H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{R})$.

DEFINITION: A first Chern class of a complex n-manifold is $c_1(\Lambda^{n,0}(M))$.

DEFINITION:
A Calabi-Yau manifold is a compact Kaehler manifold with $c_1(M, \mathbb{Z}) = 0$.

Calabi-Yau theorem

DEFINITION: Let (M, I, ω) be a Kaehler n-manifold, and $K(M) := \Lambda^{n,0}(M)$ its **canonical bundle.** We consider $K(M)$ as a holomorphic line bundle, $K(M) = \Omega^n M$. The natural Hermitian metric on $K(M)$ is written as

$$(\alpha, \alpha') \longrightarrow \frac{\alpha \wedge \overline{\alpha}'}{\omega^n}.$$

Denote by Θ_K the curvature of the Chern connection on $K(M)$. The **Ricci curvature** Ric of M is symmetric 2-form $\operatorname{Ric}(x, y) = \Theta_K(x, Iy)$.

DEFINITION: A Kähler manifold is called **Ricci-flat** if its Ricci curvature vanishes.

THEOREM: (Calabi-Yau)
Let (M, I, g) be Calabi-Yau manifold. **Then there exists a unique Ricci-flat Kaehler metric in any given Kaehler class.**

Calabi-Yau theorem and Monge-Ampère equation

REMARK: Let (M,ω) be a Kähler n-fold, and Ω a non-degenerate section of $K(M)$, Then $|\Omega|^2 = \frac{\Omega\wedge\overline{\Omega}}{\omega^n}$. If ω_1 is a new Kaehler metric on (M,I), h, h_1 the associated metrics on $K(M)$, then $\frac{h}{h_1} = \frac{\omega_1^n}{\omega^n}$.

COROLLARY: A metric $\omega_1 = \omega + \partial\overline{\partial}\varphi$ **is Ricci-flat if and only if** $(\omega + \partial\overline{\partial}\varphi)^n = \omega^n e^f$**,** where $-2\partial\overline{\partial}f = \Theta_{K,\omega}$.

Proof: For such f, φ, one has $\log\frac{h}{h_1} = -f$. This gives

$$\Theta_{K,\omega_1} = \Theta_{K,\omega} + \partial\overline{\partial}\frac{h}{h_1} = \Theta_{K,\omega} - 2\partial\overline{\partial}f = 0.$$

■

THEOREM: (Calabi-Yau) Let (M,ω) be a compact Kaehler n-manifold, and f any smooth function. **Then there exists a unique up to a constant function** φ such that $(\omega+\sqrt{-1}\,\partial\overline{\partial}\varphi)^n = Ae^f\omega^n$, where A is a positive constant obtained from the formula $\int_M Ae^f\omega^n = \int_M \omega^n$.

REMARK:

$$(\omega + \sqrt{-1}\,\partial\overline{\partial}\varphi)^n = Ae^f\omega^n,$$

is called **the Monge-Ampere equation.**

Uniqueness of solutions of complex Monge-Ampere equation

PROPOSITION: (Calabi) **A complex Monge-Ampere equation has at most one solution,** up to a constant.

Proof. Step 1: Let ω_1, ω_2 be solutions of Monge-Ampere equation. Then $\omega_1^n = \omega_2^n$. By construction, one has $\omega_2 = \omega_1 + \sqrt{-1}\,\partial\bar\partial\psi$. **We need to show $\psi = const$.**

Step 2: $\omega_2 = \omega_1 + \sqrt{-1}\,\partial\bar\partial\psi$ gives

$$0 = (\omega_1 + \sqrt{-1}\,\partial\bar\partial\psi)^n - \omega_1^n = \sqrt{-1}\,\partial\bar\partial\psi \wedge \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}.$$

Step 3: Let $P := \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}$. This is a positive $(n-1, n-1)$-form. **There exists a Hermitian form ω_3 on M such that $\omega_3^{n-1} = P$.**

Step 4: Since $\sqrt{-1}\,\partial\bar\partial\psi \wedge P = 0$, this gives $\psi\partial\bar\partial\psi \wedge P = 0$. Stokes' formula implies

$$0 = \int_M \psi \wedge \partial\bar\partial\psi \wedge P = -\int_M \partial\psi \wedge \bar\partial\psi \wedge P = -\int_M |\partial\psi|_3^2 \omega_3^n.$$

where $|\cdot|_3$ is the metric associated to ω_3. **Therefore $\bar\partial\psi = 0$.** ■

Exercises

PROBLEM: Suppose that $\dim_{\mathbb{R}} M = 2$. Prove that any almost complex structure on M is integrable.

PROBLEM: Construct a non-integrable almost complex structure on a manifold M with $\dim_{\mathbb{R}} M = 4$.

PROBLEM: Let (M, I) be a smooth almost complex manifold equipped with a transitive action of a group G. Assume that I is G-invariant (such a manifold is called **homogeneous**). Assume, moreover, that for some $x \in M$ there exists $\tau_x \in G$ fixing x. Consider the action of τ_x on T_xM; denote this operator by τ.

1. Suppose that $\tau = \lambda \operatorname{Id}$, where $\lambda \in \mathbb{R}$. Prove that for all $\lambda \neq 1$, the almost complex structure I is integrable.

2. Construct examples of such (M, I), G and τ_x for each $\lambda \in \mathbb{R}$.

3. Construct a homogeneous almost complex manifold which is not integrable.

4. Suppose that τ is not a scalar, but all its eigenvalues α_i satisfy $9 < |\alpha_i| < 10$. Prove that the almost complex structure I is integrable.

Please bring these assignments in writing to the next lecture (Monday, 16.07.2012).